Math. Program., Ser. A (2011) 126:351–363
DOI 10.1007/s10107-009-0282-9

FULL LENGTH PAPER

# Lifting for conic mixed-integer programming

**Alper Atamtürk · Vishnu Narayanan**

Received: 13 March 2008 / Accepted: 28 January 2009 / Published online: 19 May 2009
© The Author(s) 2009. This article is published with open access at Springerlink.com

**Abstract** Lifting is a procedure for deriving valid inequalities for mixed-integer sets from valid inequalities for suitable restrictions of those sets. Lifting has been shown to be very effective in developing strong valid inequalities for linear integer programming and it has been successfully used to solve such problems with branch-and-cut algorithms. Here we generalize the theory of lifting to conic integer programming, i.e., integer programs with conic constraints. We show how to derive conic valid inequalities for a conic integer program from conic inequalities valid for its lower-dimensional restrictions. In order to simplify the computations, we also discuss sequence-independent lifting for conic integer programs. When the cones are restricted to nonnegative orthants, conic lifting reduces to the lifting for linear integer programming as one may expect.

**Keywords** Valid inequalities · Conic optimization · Integer programming

**Mathematics Subject Classification (2000)** 90C11 · 90C25 · 90C57

---

This research is supported, in part, by the National Science Foundation Grant 0700203: Conic Integer Programming. A. Atamtürk is grateful to the hospitality of the Georgia Institute of Technology, where part of this research was conducted.

---

A. Atamtürk (✉)
Industrial Engineering and Operations Research Department, University of California,
4141 Etcheverry Hall MC 1777, Berkeley, CA 94720-1777, USA
e-mail: atamturk@berkeley.edu

V. Narayanan
Industrial Engineering and Operations Research, Indian Institute of Technology Bombay,
Powai, Mumbai 400076, India
e-mail: vishnu@iitb.ac.in

## 1 Introduction

A conic mixed-integer program is an optimization problem of the form

$$\min \ c'x \ : \ b - Ax \in \mathcal{C}, \quad x \in \mathbb{Z}^n \times \mathbb{R}^q, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\mathcal{C} \subseteq \mathbb{R}^m$ is a closed, convex, pointed cone with nonempty interior. For instance, if $\mathcal{C}$ equals the *linear cone* $\mathbb{R}^m_+$, then $b - Ax \in \mathcal{C}$ reduces to the system of linear inequalities $Ax \leq b$. Therefore, a conic integer program is a natural generalization of a linear integer program obtained by replacing $\mathbb{R}^m_+$ with a more general cone. Throughout we assume that the feasible region of (1) is bounded and explicit bounds on the variables, if any, are included in the constraints.

A particularly interesting (nonlinear) conic constraint is the *conic quadratic* (or *second-order conic*) constraint

$$\| Ax - b \|_2 \leq d'x - e,$$

which includes a convex quadratic constraint as a special case. Many engineering and science problems, e.g., signal processing, portfolio optimization, support vector machines, are formulated with conic quadratic constraints [10]. We refer the reader to Ben-Tal and Nemirovski [6] for in-depth lecture notes on continuous conic optimization.

Due to the growing demand for solving conic quadratic integer programs, commercial optimization packages, such as CPLEX and Mosek, already offer branch-and-bound solvers for conic quadratic integer programs. However, these solvers are currently far from being able to handle large-scale conic quadratic integer programs, and much work is needed to make them nearly as effective as their counterparts for linear integer programs.

Although there is a wealth of literature on linear and quadratic integer programming, and more generally on global optimization, research on conic integer programming is so far limited. Çezik and Iyengar [8] give Chvátal–Gomory and disjunctive cuts for conic integer and conic 0–1 mixed programs. Atamtürk and Narayanan [4] describe mixed-integer rounding inequalities for conic mixed-integer programming. Vielma et al. [16] develop a branch-and-bound algorithm based on the polyhedral approximation of conic quadratic programs due to Ben-Tal and Nemirovski [7]. Aktürk et al. [1] give strong conic quadratic reformulations for mixed 0–1 problems with a separable convex objective.

Strong relaxations obtained by adding valid inequalities to formulations are crucial for solving linear integer programs with branch-and-bound methods. *Lifting* is an effective procedure for deriving such valid inequalities from simpler restrictions of the feasible set of solutions. Since 1970s lifting has been studied extensively and has become one of the most effective techniques for developing valid inequalities for *linear* integer programs. We list references [2,3,9,11–14,17,18] as a small sample of work on lifting for linear mixed-integer programming.

**Contributions** The success of lifting as an effective method for generating valid inequalities for linear integer programming is our motivation for generalizing it to *conic* mixed-integer programming in an effort to derive *conic valid inequalities* for this problem class. This generalization is achieved by working with generalized (conic) inequalities [6] and extending *lifting functions* for the linear case to *lifting sets* for conic lifting. When restricted to linear cones, conic lifting reduces to lifting for linear mixed-integer programming as one may expect.

In a recent paper Richard and Tawarmalani [15] develop an interesting lifting procedure for nonlinear programming (NLP). They lift linear inequalities valid for a low-dimensional restriction of an NLP to linear inequalities in the original space. Atamtürk and Narayanan [5] use lifting to derive linear inequalities for a conic quadratic binary knapsack set as the convex hull is a polytope in this case. Here we describe a lifting approach for conic integer programming that *lifts a valid conic constraint* in a low-dimensional restriction *to a valid conic constraint* for the original conic mixed-integer set. Keeping the lifted constraint conic responds to a practical concern: if the continuous relaxations are solved with a conic solver, e.g., an SOCP solver, we would like to keep the valid inequalities added to the formulation conic. Moreover, maintaining the conic structure as an inequality is lifted, makes it easy to characterize the lifting sets and appropriate bounds for them.

**Outline** In Sect. 2 we develop the theory of conic lifting and describe how to derive lifted conic valid inequalities for conic mixed-integer programming by sequentially extending a low dimensional conic inequality. In Sect. 3 we discuss when lifting can be done independent of a lifting sequence so as to simplify the computations. In Sect. 4 we describe an application of sequence-independent conic lifting to a conic quadratic 0–1 set for illustration. In Sect. 5 we discuss the connections to the special case of linear mixed-integer programming. Finally, in Sect. 6 we conclude with a few final remarks.

Throughout the paper for two sets $A$ and $B$ in $\mathbb{R}^m$, $A + B$ denotes their Minkowski sum.

## 2 Conic lifting

Let us consider a conic mixed-integer set

$$S^n(b) := \left\{ (x^0, \dots, x^n) \in X^0 \times \cdots \times X^n \ : \ b - \sum_{i=0}^{n} A^i x^i \in \mathcal{C} \right\},$$

where $A^i \in \mathbb{R}^{m \times n_i}$, $b \in \mathbb{R}^m$, and $\mathcal{C} \subseteq \mathbb{R}^m$ is a *proper* cone, i.e., closed, convex, pointed cone with nonempty interior, and each $X^i$ is a mixed-integer set in $\mathbb{R}^{n_i}$. We assume that $S^n(b)$ is bounded.

We are interested in deriving conic valid inequalities for $S^n(b)$, from conic inequalities valid for restrictions of $S^n(b)$. As indexing is arbitrary, consider a nonempty restriction $S^0(b - \sum_{i=1}^n A^i \bar{x}^i)$ by fixing $x^i = \bar{x}^i \in X^i$ for all $i = 1, \dots, n$. In order to simplify the notation, by replacing variables $x^i$ with $x^i - \bar{x}^i$ and updating $b$ as $b - \sum_{i=1}^n A^i \bar{x}^i$, we may assume without loss of generality that $\bar{x}^i = \mathbf{0}$ for $i = 1, \ldots, n$ and hence the restriction is $S^0(b)$.

Let $\mathcal{K} \subseteq \mathbb{R}^p$ be again a proper cone and consider a valid inequality

$$h - F^0 x^0 \in \mathcal{K} \tag{2}$$

for $S^0(b)$, where $h$ and $F^0$ are matrices of appropriate dimensions. Note that $\mathcal{C}$ and $\mathcal{K}$ may be different cones. For instance, $\mathcal{K}$ may be the cone of positive semidefinite matrices and $\mathcal{C}$ may be the linear cone, or $\mathcal{K}$ may be the linear cone and $\mathcal{C}$ may be the conic quadratic cone.

Starting from valid inequality (2) for the restriction $S^0(b)$, our goal is to iteratively compute (matrices) $F^1, \ldots, F^i$ such that *the lifted inequality*

$$h - \sum_{j=0}^{i} F^j x^j \in \mathcal{K} \tag{3}$$

is valid for $S^i(b)$ for $i = 1, \ldots, n$. Toward this end, for conic inequality (3) we define a *lifting set* below. We use $(\mathcal{F}^i, h)$ to denote the intermediate lifted conic inequality (3).

**Definition 1** For $v \in \mathbb{R}^m$ and $i \in \{0, \ldots, n\}$, let *the lifting set* corresponding to inequality (3) be

$$\Phi_i(v) := \left\{ d \in \mathbb{R}^p : h - \sum_{j=0}^{i} F^j x^j - d \in \mathcal{K} \text{ for all } (x^0, \ldots, x^i) \in S^i(b - v) \right\}.$$

The lifting set $\Phi_i(v)$ is a subset of $\mathbb{R}^p$ parametrized by $v \in \mathbb{R}^m$. Observe that if $S^i(b - v) = \emptyset$, then $\Phi_i(v) = \mathbb{R}^p$. The lifting set $\Phi_i$ is used for computing $F^{i+1}$ as shown in the sequel.

**Proposition 1** *For $i = 0, \ldots, n$ and $v \in \mathbb{R}^m$, $\Phi_i(v)$ is a nonempty, closed, convex set with recession cone $-\mathcal{K}$.*

*Proof* We first show that $\Phi_i(v)$ is nonempty. As $S^i(b)$ is bounded, so is $S^i(b - v)$. Then for $y \in \mathbb{R}^p$, $z(y) := \max \left\{ \sum_{j=0}^{i} y' F^j x^j : x \in S^i(b - v) \right\} < \infty$. Let $z^* := \sup\{z(y) : y \in \mathcal{K}^*, ||y|| = 1\}$ and $g \in \mathcal{K}$ such that $y'g \geq z^*$ for all $y \in \mathcal{K}^*$ and $||y|| = 1$, where $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$. Then, by scaling $y$ we have $y'(g - \sum_{j=0}^{i} F^j x^j) \geq 0$ for all $y \in \mathcal{K}^*$ and all $(x^0, \ldots, x^i) \in S^i(b - v)$, implying validity of $g - \sum_{j=0}^{i} F^j x^j \in \mathcal{K}$ for $S^i(b - v)$. Letting $d = h - g$, we see that $d \in \Phi_i(v)$.

Closedness and convexity of $\Phi_i(v)$ follows from closedness and convexity of $\mathcal{K}$. Finally, for any $t \in -\mathcal{K}$, $\alpha \in \mathbb{R}_+$ and $d \in \Phi_i(v)$, it is clear that $d + \alpha t \in \Phi_i(v)$. Conversely, let $t$ be a ray of the recession cone. Then, for any $d \in \Phi_i(v)$ and $\alpha \in \mathbb{R}_+$, $d + \alpha t \in \Phi_i(v)$, i.e., $h - \sum_{j=0}^{i} F^j x^j - (d + \alpha t) \in \mathcal{K}$ for all $(x^0, \ldots, x^i) \in S^i(b - v)$. Taking inner product with $y \in \mathcal{K}^*$, we see that $\alpha t' y \leq (h - \sum_{j=0}^{i} F^j x^j - d)' y$ for all $\alpha \in \mathbb{R}_+$, which implies that $t' y \leq 0$. Hence, $t \in -\mathcal{K}$. □

**Proposition 2** $0 \in \Phi_i(0)$ *for all* $i = 0, \dots, n$.

*Proof* Immediate from the validity of (3) for $S^i(b)$. □

**Proposition 3** $\Phi_{i+1}(v) \subseteq \Phi_i(v)$ *for all* $v \in \mathbb{R}^m$ *and* $i = 0, \dots, n-1$.

*Proof* Follows from $0 \in X^i$ and $S^i(b) \subseteq S^{i+1}(b)$ for all $i = 0, \dots, n-1$. □

The next proposition describes the set of valid lifting matrices $F^{i+1}$ based on the lifting set $\Phi_i$ corresponding to the intermediate lifted inequality $(\mathcal{F}^i, h)$.

**Proposition 4** *Inequality* $(\mathcal{F}^{i+1}, h)$ *is valid for* $S^{i+1}(b)$ *if and only if* $F^{i+1}t \in \Phi_i(A^{i+1}t)$ *for all* $t \in X^{i+1}$ *and* $i = 0, \dots, n-1$.

*Proof* Suppose that the condition is satisfied. Then, it immediately follows from the definition of $\Phi_i$ that $(\mathcal{F}^{i+1}, h)$ is valid for $S^{i+1}(b)$. Conversely, suppose that there exists some $\bar{x}^{i+1} \in X^{i+1}$ such that $F^{i+1}\bar{x}^{i+1} \notin \Phi_i(A^{i+1}\bar{x}^{i+1})$. Then, there exists an $(\bar{x}^0, \dots, \bar{x}^i) \in S^i(b - A^{i+1}\bar{x}^{i+1})$ such that $h - \sum_{j=0}^{i} F^j \bar{x}^j - F^{i+1}\bar{x}^{i+1} \notin \mathcal{K}$. However, $(\bar{x}^1, \dots, \bar{x}^{i+1}) \in S^{i+1}(b)$, which implies that $(\mathcal{F}^{i+1}, h)$ is not valid for $S^{i+1}(b)$. □

A recursive relationship between lifting sets, which is used in the next section, follows from Definition 1.

**Proposition 5** *Given* $\Phi_i$ *and* $F^{i+1}$ *for* $i = 0, \dots, n-1$, $\Phi_{i+1}$ *can be computed recursively as*

$$\Phi_{i+1}(v) = \bigcap_{t \in X^{i+1}} \left( \Phi_i(v + A^{i+1}t) - F^{i+1}t \right), \quad v \in \mathbb{R}^m.$$

*Proof* From the definition of $\Phi$, we have

$$\begin{aligned}
\Phi_{i+1}(v) &= \left\{ d : h - \sum_{j=0}^{i+1} F^j x^j - d \in \mathcal{K}, \forall x \in S^{i+1}(b - v) \right\} \\
&= \bigcap_{t \in X^{i+1}} \left\{ d : h - \sum_{j=1}^{i} F^j x^j - (F^{i+1}t + d) \in \mathcal{K}, \forall x \in S^i \left( b - v - A^{i+1}t \right) \right\} \\
&= \bigcap_{t \in X^{i+1}} \left( \Phi_i(v + A^{i+1}t) - F^{i+1}t \right).
\end{aligned}$$

□

## 3 Sequence-independent lifting

Lifting, in general, is sequence-dependent; that is, the order in which variables $x^i$, $i = 1, \dots, n$ are introduced to the lifted inequality, can change $\Phi_i$ and, consequently, the lifted inequality. In the previous section, we have seen that $\Phi_{i+1}(v) \subseteq \Phi_i(v)$ holds in general. If the converse is also true for all $i = 0, \dots, n-1$, then $\Phi_i(v) = \Phi_0(v)$ for all $v$ and $i = 1, \dots, n$, and thus the order of lifting does not matter. If this condition is satisfied, then lifting is said to be *sequence-independent*.

Recognizing sequence independence of lifting is computationally desirable, as in this case, it is sufficient to compute only the first lifting set $\Phi_0$. Here we give a sufficient condition for sequence-independent lifting and show how to lift an inequality efficiently by a superadditive subset (see Definition 2 below) of the lifting set $\Phi_0$. Sequence-independence of lifting by superadditive functions have been shown for linear 0–1 and mixed 0–1 programs by Wolsey [18] and Gu et al. [9]. The results below generalize of the ones in Atamtürk [3] for the linear mixed-integer programming case.

**Definition 2** The parametrized set $\Phi(v)$ is called *superadditive* on $D \subseteq \mathbb{R}^m$ if for all $u, v, u+v \in D$, we have $\Phi(u) + \Phi(v) \subseteq \Phi(u+v)$, i.e., $\alpha + \beta \in \Phi(u+v)$ for all $\alpha \in \Phi(u)$, $\beta \in \Phi(v)$.

**Lemma 1** *The parametrized set*

$$\Psi(v) := \bigcap_{w \in \mathbb{R}^m} \bigcap_{\pi \in \Phi_0(w)} (\Phi_0(v+w) - \pi), \tag{4}$$

*is superadditive on* $\mathbb{R}^m$.

*Proof* If either $\Psi(u) = \emptyset$ or $\Psi(v) = \emptyset$, then $\Psi(u) + \Psi(v) \subseteq \Psi(u+v)$ holds trivially. Otherwise, suppose for contradiction that $\alpha \in \Psi(u)$ and $\beta \in \Psi(v)$, but $\alpha + \beta \notin \Psi(u+v)$. Then, there exist a $w$ and $\pi \in \Phi_0(w)$ such that $\alpha+\beta+\pi \notin \Phi_0(u+v+w)$. However, because $\beta \in \Psi(v)$, we must have $\beta + \pi \in \Phi_0(v+w)$, which, together with $\alpha \notin \Phi_0(u+v+w) - (\beta+\pi)$, implies that $\alpha \notin \Psi(u)$. A contradiction with the choice of $\alpha$. □

The next theorem implies that $\Psi$ is contained in any lifting set, *independent* of the lifting order.

**Theorem 1** *The parametrized set* $\Psi$ *is a subset of the last lifting set; that is,* $\Psi(v) \subseteq \Phi_n(v)$ *for all* $v \in \mathbb{R}^m$.

*Proof* It follows from Proposition 5 that

$$\Phi_n(v) = \bigcap_{x \in X^1 \times \cdots \times X^n} \left\{ \Phi_0\left( v + \sum_{i=1}^n A^i x^i \right) - \sum_{i=1}^n F^i x^i \right\} \supseteq \Psi(v),$$

where the inclusion follows from $\sum_{i=1}^n F^i x^i \in \Phi_0\left(\sum_{i=1}^n A^i x^i\right)$ by validity of the lifted inequality $(\mathcal{F}^n, h)$ and from the definition of $\Psi$ by taking $w = \sum_{i=1}^n A^i x^i$ and $\pi = \sum_{i=1}^n F^i x^i$. □

**Corollary 1** *The parametrized set* $\Psi$ *is a subset of all lifting sets; that is,* $\Psi(v) \subseteq \Phi_n(v) \subseteq \cdots \subseteq \Phi_0(v)$ *for all* $v \in \mathbb{R}^m$.

*Proof* Follows from Proposition 3 and Theorem 1. □

**Theorem 2** *$\Psi = \Phi_0$ if and only if $\Phi_0$ is superadditive on $\mathbb{R}^m$.*

*Proof* If $\Psi = \Phi_0$, from Theorem 1 $\Phi_0$ is superadditive. Conversely, if $\Phi_0$ is superadditive, then $\Phi_0(v) \subseteq \Phi_0(v+w) - \pi$ for all $\pi \in \Phi_0(w)$. Hence, $\Phi_0(v) \subseteq \cap_{\pi \in \Phi_0(w)} (\Phi_0(v+w) - \pi)$. Since this is true for all $w$, we can take intersection over all $w$ to see that $\Phi_0(v) \subseteq \Psi(v)$. But from Corollary 1 $\Phi_0(v) \supseteq \Psi(v)$ holds as well. □

**Corollary 2** *If $\Phi_0$ is superadditive, then $\Psi = \Phi_0 = \Phi_1 = \cdots = \Phi_n$, i.e., lifting is sequence-independent.*

The next proposition states that even if $\Phi_0$ is not superadditive, in general any superadditive subset of $\Phi_0$ may be employed for sequence-independent lifting of $(\mathcal{F}^0, h)$.

**Theorem 3** *If $\Omega(v) \subseteq \Phi_0(v)$ for all $v \in \mathbb{R}^m$ and $\Omega$ is superadditive on $\mathbb{R}^m$, then $(\mathcal{F}^n, h)$ is a lifted valid inequality for $S^n(b)$ whenever $F^i t \in \Omega(A^i t)$ for all $t \in X^i$ and $i = 1, \ldots, n$.*

*Proof* From the assumptions of the proposition, for any $x \in S^n(b)$, we have

$$\sum_{i=1}^{n} \Omega(A^i x^i) \subseteq \Omega\left(\sum_{i=1}^{n} A^i x^i\right) \subseteq \Phi_0\left(\sum_{i=1}^{n} A^i x^i\right).$$

Then $\sum_{i=1}^{n} F^i x^i \in \Phi_0\left(\sum_{i=1}^{n} A^i x^i\right)$ and thus $h - \sum_{i=0}^{n} F^i x^i \in \mathcal{K}$. □

**Corollary 3** *If $\Phi_0$ is superadditive on $\mathbb{R}^m$, then $(\mathcal{F}^n, h)$ is a lifted valid inequality for $S^n(b)$ whenever $F^i t \in \Phi_0(A^i t)$ for all $t \in X^i$ and $i = 1, \ldots, n$.*

The next two propositions show that we can assume, without loss of generality, that $\Omega(v)$ is convex and $0 \in \Omega(0)$ for a superadditive parametrized set $\Omega$ satisfying $\Omega(v) \subseteq \Phi_0(v)$ for all $v$.

**Proposition 6** *Suppose that $\Omega(v) \subseteq \Phi_0(v)$ for all $v$ and $\Omega$ is superadditive. Then, there exists a superadditive parametrized set $\Omega'$ such that $\Omega'(v)$ is convex and $\Omega(v) \subseteq \Omega'(v) \subseteq \Phi_0(v)$ for all $v$.*

*Proof* Let $\Omega'(v) = \text{conv}(\Omega(v))$. Then, $\Omega'$ is superadditive as $\text{conv}(A) + \text{conv}(B) = \text{conv}(A+B)$ for any $A, B \subseteq \mathbb{R}^m$ and $\Omega'(v) \subseteq \Phi_0(v)$ by convexity of $\Phi_0(v)$ for all $v$. □

**Proposition 7** *Suppose that $\Omega(v)$ is convex and $\Omega(v) \subseteq \Phi_0(v)$ for all $v$ and $\Omega$ is superadditive. Then, there exists a superadditive parametrized set $\Omega'$ such that*

(1) *$\Omega'(v)$ is convex for all $v$,*
(2) *$\Omega(v) \subseteq \Omega'(v) \subseteq \Phi_0(v)$ for all $v$, and*
(3) *$0 \in \Omega'(0)$.*

*Proof* As $\Omega$ is superadditive, $\Omega(v) + \Omega(0) \subseteq \Omega(v)$ for all $v$. Hence, for all $\pi \in \Omega(v)$ and all $d \in \Omega(0)$, we have $\pi + d \in \Omega(v)$. Repeating the argument $k$ times ($k \in \mathbb{Z}_+$), we see that $\pi + kd \in \Omega(v)$ for all $k \in \mathbb{Z}_+$. Thus, $\Omega(0)$ is contained in the recession cone of $\Omega(v)$ for all $v$.

Now, let $\Omega'$ be defined as

$$\Omega'(v) := \begin{cases} \Omega(v) & \text{if } \ v \neq 0, \\ \mathcal{K}' & \text{if } \ v = 0, \end{cases}$$

where $\mathcal{K}'$ is the cone generated by $\Omega(0) \cup \{0\}$. It is easily seen that $\mathcal{K}'$ is also contained in the recession cone of $\Omega(v)$ for all $v$. By construction, $0 \in \Omega'(0)$ and $\Omega(v) \subseteq \Omega'(v) \subseteq \Phi_0(v)$ for all $v$. We now show that $\Omega'$ is superadditive.

It is clear that if $u, v \neq 0$ or if $u = v = 0$, $\Omega'(u) + \Omega'(v) \subseteq \Omega'(u+v)$. Now suppose that $u = 0$, $v \neq 0$. Then, $\Omega'(u) + \Omega'(v) = \mathcal{K}' + \Omega(v) = \Omega(v) = \Omega'(u+v)$. Hence, the result. □

Since in both propositions $\Omega'(v)$ contains $\Omega(v)$, any valid inequality generated using $\Omega$ can also be generated using $\Omega'$.

## 4 An illustrative application

In this section we illustrate sequence-independent conic lifting for a conic quadratic mixed 0–1 set. Recall that an $m+1$-dimensional second-order cone is defined as

$$\mathcal{Q}^{m+1} := \left\{(t, t_o) \in \mathbb{R}^m \times \mathbb{R} : ||t||_2 \leq t_o\right\}.$$

Consider the conic quadratic mixed 0–1 set

$$S = \left\{ (x, y, t) \in \{0,1\}^{n+1} \times \mathbb{R}^2 \ : \ \sqrt{\left(\sum_{i=0}^{n} a_i x_i - \beta\right)^2 + y^2} \leq t \right\}.$$

We should point out that given a high dimensional second order conic constraint

$$||Ax - b|| \leq t,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, we can construct a relaxation with $\mathcal{Q}^{2+1}$ of the form $S$, by aggregating $m-1$ terms in the square root into the nonnegative term $y^2$ in $S$. Inequalities obtained from simpler relaxations of integer programs are common in linear integer programming. For conic 0–1 programming the set $S$ plays a similar role to that of single row 0–1 knapsack relaxations in linear 0–1 programming.

If necessary, by complementing the binary variables, we assume that $a > \mathbf{0}$. Fixing $x_i = 0$, $i = 1, \ldots, n$, we arrive at the restriction

$$S^0 = \left\{ (x_0, y, t) \in \{0,1\} \times \mathbb{R}^2 \ : \ \sqrt{(x_0 - \beta)^2 + y^2} \leq t \right\},$$

where we assume without loss of generality that $a_0 = 1$. The continuous relaxation of $S^0$ has the unique extreme point $(\beta, 0, 0)$, which is fractional for $x_0$ when $0 < \beta < 1$. This fractional extreme point is cut off by the conic inequality

$$\sqrt{((2\beta - 1)x_0 - \beta)^2 + y^2} \le t. \tag{5}$$

Indeed, Atamtürk and Narayanan [4] show that adding inequality (5) to the continuous relaxation of $S^0$ is sufficient to describe $\text{conv}(S^0)$.

Our goal in this section is to lift the conic quadratic inequality (5) for $S^0$ to a conic quadratic inequality of the form

$$\sqrt{\left((2\beta - 1)x_0 + \sum_{i=1}^{n} \alpha_i x_i - \beta\right)^2 + y^2} \le t, \tag{6}$$

that is valid for $S$.

Now, letting $b = (\beta, 0, 0)'$, let us write $S$ in matrix form

$$S^n(b) = \left\{ (x, y, t) \in \{0, 1\}^{n+1} \times \mathbb{R}^2 : \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x_0 \\ y \\ t \end{bmatrix} - \sum_{i=1}^{n} \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix} x_i \in \mathcal{Q}^3 \right\}$$

as in Sect. 2. Similarly inequality (6) is also written as

$$\begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} (2\beta - 1)x_0 \\ y \\ t \end{bmatrix} - \sum_{i=1}^{n} \begin{bmatrix} \alpha_i \\ 0 \\ 0 \end{bmatrix} x_i \in \mathcal{Q}^3.$$

We will compute the lifting set $\Phi_0(v)$ corresponding to (5), where $v \in \mathbb{R}^3$ and $v_1 \ge 0$ as $a > \mathbf{0}$. Given $v$, this amounts to computing the set of all $d \in \mathbb{R}^3$ such that

$$\begin{pmatrix} \beta - (2\beta - 1)x_0 \\ -y \\ -t \end{pmatrix} - d \in \mathcal{Q}^3 \tag{7}$$

for all $(x_0, y, t) \in S^0(b - v)$. That is,

$$\Phi_0(v) = \left\{ d \in \mathbb{R}^3 : (7) \text{ holds for all } (x_0, y, t) \in S^0(b - v) \right\}.$$

In order to derive the coefficients $\alpha_i$, $i = 1, \ldots, n$, of (6), we will construct a superadditive subset $\Omega$ of the lifting set $\Phi_0$ per Theorem 3 and Propositions 6 and 7. Recall that lifting coefficients $F_i$ must satisfy $F_i t \in \Omega(A_i t)$ for $t \in \{0, 1\}$.

Because in this case,

$$F_i = \begin{bmatrix} \alpha_i \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A_i = \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix} \quad \text{for } i = 1, \ldots, n, \tag{8}$$

it is sufficient to concentrate on the first component of the lifting set. Hence, the lifting set is then $\Phi_0(v_1, 0, 0) = \lambda(v_1) - \mathcal{Q}^3$, where

$$\lambda(v_1) = \left\{ \begin{pmatrix} d_1 \\ 0 \\ 0 \end{pmatrix} : \sqrt{(\beta - d_1 - (2\beta - 1)x_0)^2 + y^2} \le t, \forall (x_0, y, t) \in S^0 \begin{pmatrix} \beta - v_1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

For computing $\lambda(v_1)$, we consider the cases $x_0 = 0$ and $x_0 = 1$ below.

*Case* $x_0 = 0 : (d_1, 0, 0) \in \lambda(v_1)$ if and only if

$$\sqrt{(\beta - d_1)^2 + y^2} \le t \quad \text{for all } (y, t) \text{ s.t. } \sqrt{(\beta - v_1)^2 + y^2} \le t.$$

This condition holds if and only if $|\beta - d_1| \le |\beta - v_1|$.

*Case* $x_0 = 1 : (d_1, 0, 0) \in \lambda(v_1)$ if and only if

$$\sqrt{(1 - \beta - d_1)^2 + y^2} \le t \quad \text{for all } (y, t) \text{ s.t. } \sqrt{(\beta - v_1 - 1)^2 + y^2} \le t,$$

which holds if and only if $|1 - \beta - d_1| \le |1 + v_1 - \beta|$.

It follows from these conditions that $\lambda(v_1)$ is a polyhedral set. We now consider the following cases:

1. $v_1 \ge \beta$: In this case, the conditions reduce to $|\beta - d_1| \le v_1 - \beta$ and $|1 - \beta + d_1| \le 1 + v_1 - \beta$; and they are satisfied if and only if $2\beta - v_1 \le d_1 \le v_1$.
2. $0 \le v_1 < \beta$: In this case, the conditions become $|\beta - d_1| \le \beta - v_1$ and $|1 - \beta + d_1| \le 1 + v_1 - \beta$; and they hold if and only if $v_1 \le d_1 \le \min\{2\beta - v_1, \}$ $2(1 - \beta) + v_1$.

   Therefore, we have

$$\lambda(v_1) = \left\{ \begin{pmatrix} d_1 \\ 0 \\ 0 \end{pmatrix} : \begin{array}{ll} v_1 \le d_1 \le \min\{2\beta - v_1, 2(1 - \beta) + v_1\} & \text{if } 0 \le v_1 < \beta \\ 2\beta - v_1 \le d_1 \le v_1 & \text{if } v_1 \ge \beta. \end{array} \right.$$

Clearly, $\lambda$ is superadditive if and only if the upper bound on $d_1$ is superadditive and the lower bound on $d_1$ is subadditive. Hence, $\lambda$ is superadditive for $v_1 \ge \beta$; however, it is not so over the entire $v_1 \ge 0$ as the upper bound for $0 \le v_1 \le \beta$ is not superadditive. Therefore, in order to construct a valid lifting set, by only picking the lower bound for $0 \le v_1 \le \beta$, we define

$$\lambda'(v) = \begin{cases} (v_1, 0, 0)' & \text{if } 0 \le v_1 < \beta, \\ \lambda(v_1) & \text{if } v_1 \ge \beta. \end{cases}$$

Consider now the parametrized set $\Omega(v) := \lambda'(v) - \mathcal{Q}^3$. Because $\lambda'(v) \subseteq \lambda(v)$, we have $\Omega(v) \subseteq \Phi_0(v)$. On the other hand, superadditivity of $\Omega$ follows from superadditivity of $\lambda'$, which follows from fact that $\varphi_1(v_1) = v_1$ is both subadditive and superadditive, and that $\varphi_2(v_1) = 2\beta - v_1$ is subadditive as $\beta > 0$. Hence, we have shown that $\Omega(v)$ is a superadditive lifting set for (5).

Now that we have a superadditive lifting set, we are ready to compute the lifting "matrices" $F_i$, $i = 1, \ldots, n$. Since $0 \in \Omega(0)$, it suffices to find $F_i \in \Omega(A_i)$ satisfying Proposition 4 only for $t = 1$. Recalling (8), we find inequalities (6) based on $\Omega$, by picking extreme values of $\lambda'(a_i)$:

$$\alpha_i \in \begin{cases} \{a_i\} & \text{if } a_i < \beta \\ \{a_i, 2\beta - a_i\} & \text{if } a_i \geq \beta \end{cases} \quad \text{for } i = 1, \ldots, n.$$

Observe that the superadditive conic lifting inequality (6) is not unique; for $a_i \geq \beta$, $\alpha_i$ may be chosen as either $a_i$ or $2\beta - a_i$. Therefore, the lifted inequalities form an exponential class. Unlike the conic MIR inequalities in Atamtürk and Narayanan [4], the lifted inequalities presented here make an explicit use of the binary variables.

*Example 1* Let $S$ be defined by $a = \mathbf{1}$ and $\beta = 1/2$; thus the conic constraint is

$$\sqrt{\left(\sum_{i=0}^{n} x_i - 1/2\right)^2 + y^2} \leq t. \tag{9}$$

Let us consider lifting the special case of inequality (5)

$$\sqrt{(-1/2)^2 + y^2} \leq t,$$

which is valid for the restriction

$$S^0 = \left\{x_0 \in \{0, 1\}, y \in \mathbb{R}, t \in \mathbb{R} : \sqrt{(x_0 - 1/2)^2 + y^2} \leq t\right\}.$$

Then, by choosing $\alpha_i = 1$ or $0$ for $i = 1, \ldots, n$, we obtain exponentially many superadditive conic lifting inequalities

$$\sqrt{\left(\sum_{i \in T} x_i - 1/2\right)^2 + y^2} \leq t \quad \text{for } T \subseteq \{1, \ldots, n\}. \tag{10}$$

Observe that for each set $T$ inequality (10) supports $\text{conv}(S)$ at points $(x, y, t)$: $(\mathbf{0}, 0, \frac{1}{2})$ and $(\sum_{i \in Y} e_i, 0, |Y| - \frac{1}{2})$ for any $\emptyset \neq Y \subseteq T$.

In contrast, the conic MIR inequality of Atamtürk and Narayanan [4] for this example is just the special case of inequality (10) with $T = \emptyset$. This example illustrates the power of the conic lifting for describing the coefficients of valid inequalities for conic MIPs.

## 5 Lifting for linear mixed-integer programming

Here we show that lifting *linear* valid inequalities for *linear* mixed-integer programs is a direct consequence of conic lifting. Let the cone $\mathcal{C} = \mathbb{R}^m_+$, so that

$$S^n(b) = \left\{ (x^0, \ldots, x^n) \in X^0 \times \cdots \times X^n \, : \, \sum_{i=0}^{n} A^i x^i \leq b \right\}.$$

With $\mathcal{K} = \mathbb{R}_+$, we have the inequality $F^0 x^0 \leq h$ valid for $S^0(b)$, where $h \in \mathbb{R}$, $F^0$ is a $1 \times n_0$ row vector. In this case, the lifting set is given by

$$\Phi_i(v) = \{\alpha \in \mathbb{R} \, : \, \alpha \leq \varphi_i(v)\},$$

where $\varphi_i(v)$ is the lifting function

$$\varphi_i(v) = \min \left\{ h - \sum_{j=0}^{i} F^j x^j \, : \, (x^0, \ldots, x^i) \in S^i(b - v) \right\}.$$

Note that $\beta \in \Phi_i(v)$ if and only if $\beta \leq \varphi_i(v)$. Consequently, Proposition 2 states that $\varphi_i(0) \geq 0$ and Proposition 3 implies that $\varphi_i(v) \geq \varphi_{i+1}(v)$ for all $i = 1, \ldots, n-1$. Suppose we have an intermediate lifted inequality $(\mathcal{F}^i, h)$ for $S^i(b)$, then inequality $(\mathcal{F}^{i+1}, h)$ is valid for $S^{i+1}(b)$ if and only if $F^{i+1} t \leq \varphi_i(A^{i+1} t)$ for all $t \in X^{i+1}$, by Proposition 4.

It is also seen that all results in Sect. 3 specialize to sequence-independent lifting in the case of linear mixed-integer programs [3,9,18]. The lifting set $\Phi_0$ is superadditive if and only if the lifting function $\varphi_0$ is superadditive. Similarly, in the linear case, superadditive subsets of $\Phi_0$ correspond to superadditive lower bounding functions of $\varphi_0$. Hence, we see that lifting in the linear case can be obtained by letting the cones $\mathcal{C}$ and $\mathcal{K}$ be nonnegative orthants of appropriate dimension.

## 6 Final remarks

We have shown how to lift a conic inequality valid for lower-dimensional restrictions of a conic mixed-integer set into valid conic inequalities for the original set. As in the linear case, superadditive lifting sets lead to sequence-independent lifting. The conic lifting approach is illustrated on a conic quadratic mixed 0–1 set. Because the coefficients are described by lifting sets rather than lifting functions, sequence independent lifting of the inequalities may lead to an exponential class of lifted inequalities as illustrated with an example. Based on the many successful applications of lifting in the linear case, it is reasonable to expect conic lifting to become an effective method for deriving conic inequalities for structured conic mixed-integer programs as well in due time.

**Acknowledgments** We are thankful to an anonymous referee for several valuable comments, especially the ones that led to Propositions 6 and 7.

**Open Access** This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

## References

1. Aktürk, M.S., Atamtürk, A., Gürel, S.: A strong conic quadratic reformulation for machine-job assignment with controllable processing times. Research Report BCOL.07.01, IEOR, University of California-Berkeley, April 2007. Oper. Res. Lett. **37**, 187–191 (2009). doi:10.1016/j.orl.2008.12.009
2. Atamtürk, A.: On the facets of mixed-integer knapsack polyhedron. Math. Program. **98** 145–175 (2003). doi:10.1007/s10107-003-0400-z
3. Atamtürk, A.: Sequence independent lifting for mixed-integer programming. Oper. Res. **52**, 487–490 (2004). doi:10.1287/opre.1030.0099
4. Atamtürk, A., Narayanan, V.: Conic mixed-integer rounding cuts. Research Report BCOL.06.03, IEOR, University of California-Berkeley, December 2006. Math. Program. (2008). doi:10.1007/s10107-008-0239-4
5. Atamtürk, A., Narayanan, V.: The submodular 0–1 knapsack polytope. Research Report BCOL.08.03, IEOR, University of California-Berkeley, June 2008. Discrete Optim. (forthcoming)
6. Ben-Tal, A., Nemirovski, A.: Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. MPS-SIAM Series on Optimization. SIAM, Philadelphia (2001a)
7. Ben-Tal, A., Nemirovski, A.: On polyhedral approximations of the second-order cone. Math. Oper. Res. **26**, 193–205 (2001b)
8. Çezik, M.T., Iyengar, G.: Cuts for mixed 0–1 conic programming. Math. Program. **104**, 179–202 (2005)
9. Gu, Z., Nemhauser, G.L., Savelsbergh, M.W.P.: Sequence independent lifting in mixed integer programming. J. Comb. Optim. **4**, 109–129 (2000)
10. Lobo, M., Vandenberghe, L., Boyd, S., Lebret, H.: Applications of second-order cone programming. Linear Algebra Appl. **284**, 193–228 (1998)
11. Louveaux, Q., Wolsey, L.A.: Lifting, superadditivity, mixed integer rounding, and single node flow sets revisited. Ann. Oper. Res. **153**, 47–77 (2007)
12. Padberg, M.W.: A note on 0–1 programming. Oper. Res. **23**, 833–837 (1979)
13. Richard, J.-P.P., de Farias, I.R., Nemhauser, G.L.: Lifted inequalities for 0–1 mixed integer programming: basic theory and algorithms. Math. Program. **98**, 89–113 (2003a)
14. Richard, J.-P.P., de Farias, I.R., Nemhauser, G.L.: Lifted inequalities for 0–1 mixed integer programming: superlinear lifting. Math. Program. **98**, 115–143 (2003b)
15. Richard, J.-P.P., Tawarmalani, M.: Lifting inequalities: a framework for generating strong cuts for nonlinear programs. Manuscript, Purdue University, March 2007. Math. Program. (forthcoming)
16. Vielma, J.P., Ahmed, S., Nemhauser, G.L.: A lifted linear programming branch-and-bound algorithm for mixed integer conic quadratic programs. INFORMS J. Comput. **20**, 438–450 (2008)
17. Wolsey, L.A.: Facets and strong valid inequalities for integer programs. Oper. Res. **24**, 367–372 (1976)
18. Wolsey, L.A.: Valid inequalities and superadditivity for 0/1 integer programs. Math. Oper. Res. **2**, 66–77 (1977)