

Lot Sizing with Inventory Bounds and Fixed Costs: Polyhedral Study and Computation

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We investigate the polyhedral structure of the lot-sizing problem with inventory bounds. We consider two models, one with linear cost on inventory, the other with linear and fixed costs on inventory. For both models, we identify facet-defining inequalities that make use of the inventory bounds explicitly and give exact separation algorithms. We also describe a linear programming formulation of the problem when the order and inventory costs satisfy the Wagner-Whitin nonspeculative property. We present computational experiments that show the effectiveness of the results in tightening the linear programming relaxations of the lot-sizing problem with inventory bounds and fixed costs.

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1. Introduction

Given the demand for an item for each time period over a finite discrete horizon, the lot-sizing problem is to determine the order and inventory quantity in each time period so that the sum of order and inventory holding costs is minimized. In this paper, we study the problem with linear and fixed costs on order as well as on inventory. Furthermore, we impose upper bounds on the inventory carried in each period. We refer to this problem as the *lot-sizing problem with bounded inventory*. Throughout we assume that there is no upper bound on the order quantity.

Motivation

Even though in almost every study on the lot-sizing problem inventory is assumed to be unbounded, for many practical applications the amount of inventory that is carried from one period to the next is bounded due to either physical constraints such as warehouse capacity or managerial policies.

Another aspect of the problem studied here, which has not received much attention in the literature, is the inventory fixed costs. Inventory fixed costs and capacities play a significant role in situations where warehousing is outsourced. The lease cost of storage space in an off-site warehouse is a fixed charge that cannot be treated as part of the standard variable holding cost, which is commonly taken as the cost of capital investment in inventory. Inventory fixed costs and capacities also arise in situations where manufacturers rent shelf space at the retailers. These fixed rental costs and limited storage capacities are considerations that cannot be overlooked.

It has been demonstrated in earlier studies (Pochet and Wolsey 1991, Wolsey 2002) that a good understanding of

the polyhedral structure of single-item lot-sizing problems can be very useful in solving more complicated problems, involving multiple products and stages. Because the single-item lot-sizing polyhedron is contained as a fundamental substructure in those problems, investigating inventory bounds and fixed costs within the lot-sizing context is meaningful from a practical point of view.

Although the polyhedron of lot sizing with inventory bounds and fixed costs is of interest in its own right, a secondary goal in this study is to gain a better understanding of the polyhedral structure of paths in capacitated fixed-charge networks. In this context, due to its simple path network representation, the lot-sizing problem with inventory bounds and fixed costs is the first natural problem to investigate.

Relevant Literature

The lot-sizing problem lies at the core of production/order and inventory-planning applications and has been studied extensively. Wagner and Whitin (1958) give an $O(n^2)$ algorithm for the lot-sizing problem with no bounds on order quantity or inventory (*uncapacitated lot-sizing problem*), where n is the number of time periods. Federgruen and Tzur (1991), Wagelmans et al. (1992), and Aggarwal and Park (1993) give $O(n \log n)$ algorithms for this problem.

The first polyhedral study of the uncapacitated lot-sizing problem is due to Bárány et al. (1984). They give a complete linear description of the convex hull of the solutions. Variants of the uncapacitated problem that include sales and safety stocks (Loparic et al. 2001), order/production lower bounds (Constantino 1998), backloging and start-ups (Pochet and Wolsey 1988, Constantino 1996, Agra and Constantino 1999), and piecewise linear concave costs (Aghezzaf and Wolsey 1994) have been studied.

The lot-sizing problem with upper bounds on production/order quantities is \mathcal{NP} -hard (Florian et al. 1971). Pochet (1988), Miller et al. (2000), Loparic et al. (2003), and Atamtürk and Muñoz (2004) give inequalities for lot sizing with order capacities and uncapacitated inventory. If order capacities are constant, the problem can be solved in polynomial time (Florian and Klein 1980, Van Hoesel and Wagelmans 1996). Leung et al. (1989) and Pochet and Wolsey (1993) describe inequalities for this case. However, a complete linear description of the convex hull of the solutions for the constant-capacity lot sizing is unknown. Atamtürk and Hochbaum (2001) study constant-capacity lot sizing with capacity acquisition and subcontracting.

Belvaux and Wolsey (2000, 2001) discuss strong formulations and a specialized branch-and-cut system for practical lot-sizing problems. Recently, van Vyve and Ortega (2003) gave a linear description of the convex hull of the solutions to uncapacitated lot sizing with inventory fixed costs. This is the only polyhedral study we are aware of that considers inventory fixed costs.

In all of the studies mentioned above, inventory is assumed to be uncapacitated. Love (1973) proposes a polynomial algorithm for the lot-sizing problem with unbounded order quantity and bounded inventory. To the best of our knowledge, the only polyhedral study on lot sizing that considers inventory capacities is Pochet and Wolsey (1994), in which the authors study lot-sizing problems under Wagner-Whitin nonspeculative cost structure. They give a linear programming formulation of lot-sizing problems with uncapacitated and constant-capacity production and inventory upper bounds.

Lot-sizing problems are special fixed-charge network flow problems. For uncapacitated fixed-charge network flows, Van Roy and Wolsey (1985) give inequalities that are based on path substructures. Ortega and Wolsey (2000) present a computational study on the performance of path and dicut inequalities in solving the uncapacitated fixed-charge network flow problem. Padberg et al. (1985), Stallaert (1997), Gu et al. (1999), and Atamtürk (2001) describe valid inequalities for capacitated fixed-charge networks based on single-node relaxations. These inequalities do not capture the path substructure of capacitated networks. Submodular inequalities of Wolsey (1989) take into account capacities, but not the fixed charges on arcs along a path.

Outline

In §2, we formally present the lot-sizing problem with inventory bounds and fixed costs and introduce the notation used in the paper. Section 3 is devoted to the special case with linear inventory costs. In §4, we study the polyhedron for linear and fixed inventory costs and extend the inequalities defined in §3 to this more general case. We give polynomial exact separation algorithms for these inequalities. In addition, we give a linear programming formulation for the lot-sizing problem with inventory bounds and fixed

costs if order and inventory costs satisfy the Wagner-Whitin nonspeculative property. Section 5 summarizes our computational experiments on testing the effectiveness of the inequalities when used as cuts for solving the problem. We conclude in §6.

2. Lot-Sizing Problem with Bounded Inventory

For a finite planning horizon n , given the demand d_t , variable order cost p_t , and fixed order cost f_t for time periods $t \in \{1, 2, \dots, n\}$; and inventory capacity u_t , variable inventory holding cost h_t , and fixed inventory holding cost g_t for $t \in \{0, 1, \dots, n\}$; the *lot-sizing problem with bounded inventory* (LSBI) is to determine the order quantity and inventory in each period so that the sum of order and inventory holding costs over the horizon is minimized. We assume that order quantity is unbounded, although given the demand, there is an implicit upper bound on the order quantity due to the inventory capacities. We let $[i, j] := \{t \in \mathbb{Z} : i \leq t \leq j\}$ throughout.

Let y_t denote the order quantity in time period t and i_t denote the inventory at the end of period t . Also, let x_t and z_t be the fixed-charge variables for order and inventory in period t , respectively. Then, LSBI can be formulated as

$$\begin{aligned}
 \text{(LSBI)} \quad & \min \sum_{t=1}^n (f_t x_t + p_t y_t + g_t z_t + h_t i_t) + g_0 z_0 + h_0 i_0 \\
 & \text{s.t. } i_{t-1} + y_t - i_t = d_t, \quad t \in [1, n], \\
 & 0 \leq i_t \leq u_t z_t, \quad t \in [0, n], \\
 & 0 \leq y_t \leq (d_t + u_t) x_t, \quad t \in [1, n], \\
 & y \in \mathbb{R}^n, \quad x \in \{0, 1\}^n, \quad i \in \mathbb{R}^{n+1}, \\
 & z \in \{0, 1\}^{n+1}.
 \end{aligned}$$

Let $d_{tl} = \sum_{j=t}^l d_j$ for $t \in [1, l]$ and $d_{tl} = 0$ for $t > l$. We let \mathcal{Q} denote the convex hull of the feasible solutions to LSBI. Observe that for the optimization problem LSBI, if desired, one may ensure that $i_0 = i_n = 0$ in an optimal solution by assigning sufficiently high values to h_0 and h_n . Therefore, for generality, we keep i_0 and i_n in the formulation for the polyhedral analysis. Throughout the paper, we assume that the data of the model consists of rational numbers and satisfy the following:

ASSUMPTION 1. $u_t > 0$ for $t \in [0, n]$.

ASSUMPTION 2. $u_{t-1} \geq d_t$ for $t \in [1, n]$.

ASSUMPTION 3. $u_{t-1} \leq d_t + u_t$ for $t \in [1, n]$.

ASSUMPTION 4. $d_t \geq 0$ for $t \in [1, n]$.

Assumptions 1 and 3 are made without loss of generality. None of the validity proofs use Assumption 2. Therefore, all inequalities in this paper are valid without Assumption 2, which is used only in facet proofs for convenience. If $u_t = 0$ for $t \in [0, n]$, then $i_t = 0$ and the problem decomposes into two subproblems for $t \in [1, n-1]$.

If $u_{t-1} < d_t$ for $t \in [1, n]$, then $x_t = 1$ in every feasible solution. If $u_{t-1} > d_t + u_t$ for $t \in [1, n]$, then u_{t-1} can be reduced to $d_t + u_t$ without changing the feasible set of solutions. Finally, $d_t < 0$ does not make sense for the lot-sizing problem.

DEFINITION 1. For a given point (y, x, i, z) in \mathcal{Q} , a consecutive sequence of time periods $[k, l]$ is called a *block* if $i_{k-1} \in \{0, u_{k-1}\}$, $i_t \in \{0, u_t\}$, and $0 < i_t < u_t$ for all $t \in [k, l-1]$.

The block definition leads to a characterization of the extreme points of \mathcal{Q} and a polynomial dynamic programming algorithm (Love 1973) for LSBI, which can be implemented in $O(n^2)$. We represent the four types of blocks by $[k, l]_{\beta}^{\alpha}$, where $\alpha \in \{0, u_{k-1}\}$ and $\beta \in \{0, u_l\}$. It follows from the network structure of the problem that (y, x, i, z) is an extreme point of \mathcal{Q} if and only if there is at most one period with positive order quantity in every block $[k, l]$. Consequently, the order quantity in such a period equals $d_{kt} + \beta - \alpha$.

EXAMPLE 1. Suppose that LSBI is given as $(d_1, \dots, d_5) = (11, 12, 13, 14, 15)$ and $(u_0, \dots, u_5) = (\infty, 30, \infty, \infty, \infty, 5)$. Then, we can strengthen the inventory bounds as $(u_0, \dots, u_5) = (41, 30, 47, 34, 20, 5)$ to satisfy Assumption 3. Figure 1 illustrates the four types of blocks in different extreme point solutions.

The linear programming (LP) relaxation of LSBI has the same block structure; however, the fixed-charge variables take the values $x_t \in \{1, y_t/(d_t + u_t)\}$ and $z_t \in \{1, i_t/u_t\}$ and are typically highly fractional. The LP extreme point solution in Figure 1(a) has $(x_1, \dots, x_5) = (1, 0, 0, 24/34, 0)$ and $(z_1, \dots, z_5) = (0, 1, 18/47, 5/34, 3/4, 0)$.

In §3, we consider the case with linear costs on inventory, i.e., the restriction of \mathcal{Q} , where inventory fixed-charge variables are one. We describe valid inequalities that cut off all extreme points of the LP relaxation with fractional order fixed-charge variables. In §4, we generalize these inequalities to incorporate the inventory fixed-charge variables as well.

Notation. We introduce the following notation, which will be used throughout the paper: For $1 \leq k \leq l \leq n$, let:

$$p = \min\{t \in [k, l] : d_{kt} > u_{k-1}\} \quad (p = l + 1 \text{ if } u_{k-1} \geq d_{kl}),$$

$$q = \min\{t \in [k, l] : d_{kt} \geq u_{k-1}\} \quad (q = l + 1 \text{ if } u_{k-1} > d_{kl}),$$

$$r = \max\{t \in [k, p] : u_t < d_{(t+1)l}\} \quad (r = k - 1 \text{ if } u_t \geq d_{(t+1)l} \text{ for all } t \in [k, p]).$$

3. Linear Inventory Costs

In this section, we address the special case of LSBI with linear holding costs. When $g = 0$, the convex hull of the feasible set of interest is $\mathcal{P} = \{(y, x, i, z) \in \mathcal{Q} : z = 1\}$ or

$$\mathcal{P} = \text{conv} \left\{ \begin{array}{ll} i_{t-1} + y_t - i_t = d_t, & t \in [1, n], \\ 0 \leq i_t \leq u_t, & t \in [0, n], \\ 0 \leq y_t \leq (d_t + u_t)x_t, & t \in [1, n], \\ y \in \mathbb{R}^n, x \in \{0, 1\}^n, i \in \mathbb{R}^{n+1}. \end{array} \right.$$

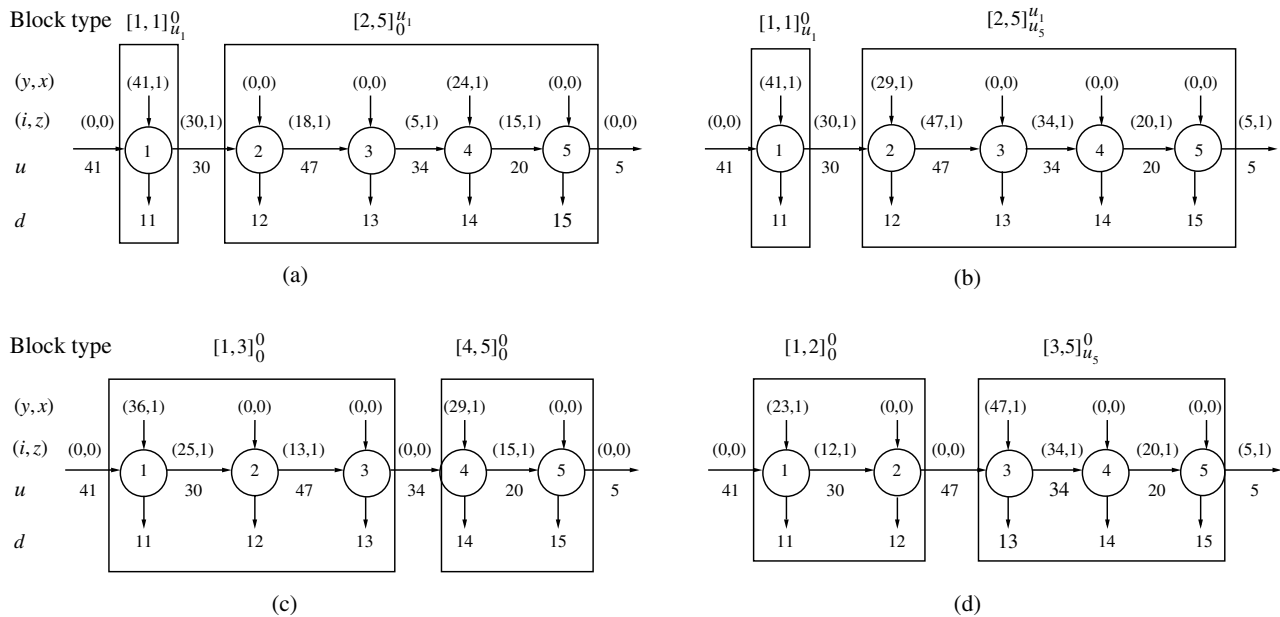
3.1. Uncapacitated Inequalities

For the special case of \mathcal{P} with no inventory capacities, Bárány et al. (1984) give the so-called (l, S) inequalities

$$\sum_{t \in S} y_t \leq \sum_{t \in S} d_{tt}x_t + i_l, \quad \text{where } S \subseteq [1, l] \text{ and } l \in [1, n]. \quad (1)$$

They show that adding inequalities (1) to the LP relaxation suffices to describe the convex hull of feasible solutions. Example 1 illustrates that in the presence of inventory

Figure 1. Four types of blocks in extreme point solutions of \mathcal{Q} .



upper bounds (l, S) inequalities may be weak and are not sufficient to describe \mathcal{P} .

EXAMPLE 1 (continued). It is easy to check that the extreme point solution of the LP relaxation with $(x_1, \dots, x_5) = (1, 29/59, 0, 0, 0)$ (in Figure 1(b)) cannot be cut off by any (l, S) inequality (1). Observe that the (l, S) inequality

$$y_1 + \dots + y_5 \leq 65x_1 + 54x_2 + 42x_3 + 29x_4 + 15x_5 + i_5,$$

with $l = 5$ and $S = [1, 5]$, can be strengthened as

$$y_1 + \dots + y_5 \leq 41x_1 + 54x_2 + 42x_3 + 29x_4 + 15x_5 + i_5$$

because $y_1 \leq 11 + 30$. However, this inequality is dominated by the variable upper-bound constraint $y_1 \leq 41x_1$ and inequality (1) with $l = 5$ and $S = [2, 5]$. Thus, if $d_t + u_t < d_{tl}$ for some $t \in S$, then inequality (1) and even the strengthening of (1),

$$\sum_{t \in S} y_t \leq \sum_{t \in S} \min\{d_{tl}, d_t + u_t\}x_t + i_l,$$

are weak.

3.2. Capacitated Inequalities

In §3.1, we have illustrated that (l, S) inequality (1) may not cut off fractional LP extreme solutions if for a block incoming or outgoing inventory is at capacity. Motivated by this observation, we obtain new inequalities by saturating incoming and outgoing inventory variables for a block.

EXAMPLE 1 (continued). To derive a strong inequality that uses inventory upper bounds, we observe that due to u_1 , it is not possible to meet the total demand $d_2 + d_3 + d_4 = 39$ from inventory i_1 . Therefore, an order must be placed in Periods 2, 3, or 4, which is stated by the inequality

$$x_2 + x_3 + x_4 \geq 1,$$

cutting off the fractional solution in Figure 1(b) that cannot be cut by the uncapacitated inequalities.

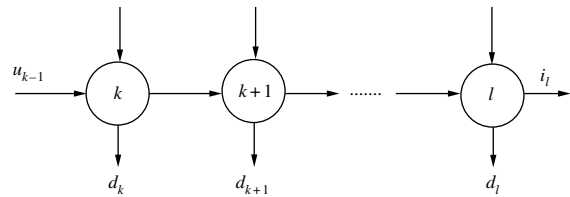
In general, for any $1 \leq k \leq l \leq n$ such that $u_{k-1} < d_{kl}$, the “cut-set type” inequality

$$\sum_{t \in [k, l]} x_t \geq 1 \tag{2}$$

is valid for \mathcal{P} . Note that if $u_{k-1} < d_{k(l-1)}$, then (2) is dominated by $\sum_{t \in [k, l-1]} x_t \geq 1$; therefore, strong inequalities among (2) must satisfy $d_{k(l-1)} \leq u_{k-1} < d_{kl}$, implying that there are only at most n strong inequalities among (2). Next, we introduce inequalities that generalize (2).

The first class of inequalities is obtained by setting the incoming inventory variable i_{k-1} for $[k, l]$ to u_{k-1} for $u_{k-1} \leq d_{kl}$ (Figure 2). Then, due to the exogenous supply u_{k-1} in time period k , the effective total demand in periods $t, t + 1, \dots, l$ is $\min\{d_{kl} - u_{k-1}, d_{tl}\}$ for $t \in [k, l]$. Also

Figure 2. Saturating incoming inventory.



observing, in this case, that the order quantity in period t cannot exceed $d_{kt} - u_{k-1} + u_t$ due to the inventory capacity u_t , we obtain the following inequalities.

For $1 \leq k \leq l \leq n$ such that $u_{k-1} \leq d_{kl}$ and $S \subseteq [k, l]$, consider the inequality

$$\begin{aligned} & i_{k-1} + \sum_{t \in S} y_t \\ & \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l. \end{aligned} \tag{3}$$

PROPOSITION 1. Inequality (3) is valid for \mathcal{P} .

PROOF. If $u_{k-1} = d_{kl}$, inequality follows from $i_{k-1} + \sum_{t \in S} y_t \leq d_{kl} + i_l = u_{k-1} + i_l$. Otherwise, let $p = \min\{t \in [k, l]: u_{k-1} < d_{kt}\}$. Observe that $d_{kl} - u_{k-1} \leq d_{tl}$ for $t \in [k, p]$ and $d_{kl} - u_{k-1} > d_{tl}$ for $t \in [p + 1, l]$. For $(y, x, i) \in \mathcal{P}$, let $b = \max\{t \in S: d_{kt} + u_t - u_{k-1} < \min\{d_{kl} - u_{k-1}, d_{tl}\} \text{ and } x_t = 1\}$ (if no such t exists, let $b = k - 1$) and let $h = \min\{t \in [b + 1, l] \cap S: x_t = 1\}$ (if no such t exists, let $h = l + 1$). If $h \leq p$, then

$$\begin{aligned} & i_{k-1} + \sum_{t \in S} y_t \\ & \leq d_{kl} + i_l = d_{kb} + d_{(b+1)l} + u_b - u_b + u_{k-1} - u_{k-1} + i_l \\ & \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l. \end{aligned}$$

The last inequality follows because $x_h = 1$, $d_{kl} - u_{k-1} \leq d_{hl}$ for $h \leq p$, $d_{(b+1)l} - u_b \leq d_{kl} - u_{k-1}$ (by Assumption 3), $d_{kl} - u_{k-1} \geq 0$, and $d_{kt} + u_t - u_{k-1} \geq 0$ (by Assumption 3).

On the other hand, if $h > p$, then

$$\begin{aligned} & i_{k-1} + \sum_{t \in S} y_t \\ & \leq d_{kb} + u_b + d_{hl} + i_l + u_{k-1} - u_{k-1} \\ & \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l. \quad \square \end{aligned}$$

REMARK 1. Note that Assumption 2 is not used to prove the validity of inequality (3). If $u_{k-1} < d_k$ for some $k \in [1, n]$, then we write inequality (3) for $k = l$ and $S = \{k\}$ as

$$i_{k-1} + y_k \leq u_{k-1} + (d_k - u_{k-1})x_k + i_k.$$

Adding the flow balance equality $d_k + i_k = y_k + i_{k-1}$, we get $d_k - u_{k-1} \leq (d_k - u_{k-1})x_k$ or $x_k \geq 1$.

REMARK 2. If there exists $j \in S \cap [p + 1, l]$ with $u_j < d_{(j+1)l}$, then inequality (3) can be strengthened as

$$i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}, d_t + u_t\}x_t + i_l. \quad (4)$$

However, inequality (4) is weak, because it is dominated by

$$i_{k-1} + \sum_{t \in S \setminus \{j\}} y_t \leq u_{k-1} + \sum_{t \in S \setminus \{j\}} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l,$$

and the constraint $y_j \leq (d_j + u_j)x_j$. It follows that $u_j \geq d_{(j+1)l}$ for $j \in [p + 1, l]$ is a necessary facet condition for inequality (3); see Proposition 13.

REMARK 3. We note that inequality (3) dominates inequality (2). To see this, suppose that $u_{k-1} < d_{kl}$, hence, inequality (2) is valid for \mathcal{P} . Rewriting inequality (3) for $S = [k, l]$ by subtracting the aggregate flow balance equality

$$i_{k-1} + \sum_{t \in [k, l]} y_t = d_{kl} + i_l,$$

we obtain

$$\sum_{t \in [k, l]} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t \geq d_{kl} - u_{k-1},$$

which is at least as strong as inequality (2).

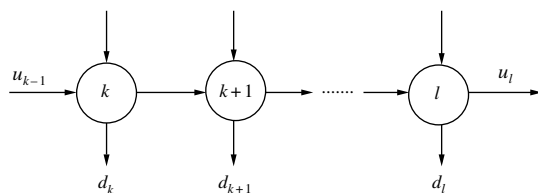
The next class of inequalities is obtained by saturating incoming as well as outgoing inventory variables i_{k-1} and i_l for a block $[k, l]$ (Figure 3). Then, due to the exogenous supply in time period k , the effective total demand in periods $t, t + 1, \dots, l$ is $\min\{d_{kl} - u_{k-1} + u_l, d_{tl} + u_l\}$ for $t \in [k, l]$. However, observing in this case that the order quantity in period t cannot exceed $d_{kt} - u_{k-1} + u_t$ due to the inventory capacity u_t , and that $d_{kt} - u_{k-1} + u_t \leq d_{kl} - u_{k-1} + u_l$ by Assumption 3, we obtain the following inequalities.

For $k \geq 1$ and $S \subseteq [k, n]$, consider the inequality

$$i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} (d_{kt} - u_{k-1} + u_t)x_t. \quad (5)$$

PROPOSITION 2. Inequality (5) is valid for \mathcal{P} .

Figure 3. Saturating incoming and outgoing inventory.



PROOF. Let $(y, x, i) \in \mathcal{P}$. If $x_t = 0$ for all $t \in S$, then inequality is trivially valid. Otherwise, let $\omega = \max\{t \in S: x_t = 1\}$. Then,

$$i_{k-1} + \sum_{t \in S} y_t \leq d_{k\omega} + u_\omega + u_{k-1} - u_{k-1} \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t. \quad \square$$

REMARK 4. Suppose that $S \subseteq [k, l]$ for $l \geq k$. Then, inequality

$$i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{tl} + u_l\}x_t \quad (6)$$

is dominated by inequality (5) with $S' = S \cap [k, q]$, where $q = \min\{t \in [k, l]: d_{kt} \geq u_{k-1}\}$ and inequality $y_t \leq (d_t + u_t)x_t$ for $t \in S \setminus S'$ because $d_t + u_t \leq d_{tl} + u_l$ by Assumption 3. It follows that $S \subseteq [k, q]$ is a necessary facet condition for Inequality (5); see Proposition 14.

In the appendix we study the strength of inequalities (1), (3), and (5) with respect to \mathcal{P} .

EXAMPLE 1 (continued). For $[k, l] = [2, 5]$, we have $p = q = 4$. Then, all facet-defining inequalities (3) of \mathcal{P} for $[k, l] = [2, 5]$ with $|S| \leq 2$ are

$$\begin{aligned} i_1 + y_2 &\leq 30 + 24x_2 + i_5, \\ i_1 + y_3 &\leq 30 + 24x_3 + i_5, \\ i_1 + y_4 &\leq 30 + 24x_4 + i_5, \\ i_1 + y_2 + y_3 &\leq 30 + 24x_2 + 24x_3 + i_5, \\ i_1 + y_2 + y_4 &\leq 30 + 24x_2 + 24x_4 + i_5, \\ i_1 + y_3 + y_4 &\leq 30 + 24x_3 + 24x_4 + i_5, \\ i_1 + y_2 + y_5 &\leq 30 + 24x_2 + 15x_5 + i_5, \\ i_1 + y_3 + y_5 &\leq 30 + 24x_3 + 15x_5 + i_5, \\ i_1 + y_4 + y_5 &\leq 30 + 24x_4 + 15x_5 + i_5. \end{aligned}$$

On the other hand, all facet-defining inequalities (5) of \mathcal{P} for $[k, l] = [2, 5]$ are

$$\begin{aligned} i_1 &\leq 30, \\ i_1 + y_2 &\leq 30 + 29x_2, \\ i_1 + y_3 &\leq 30 + 29x_3, \\ i_1 + y_4 &\leq 30 + 29x_4, \\ i_1 + y_2 + y_3 &\leq 30 + 29x_2 + 29x_3, \\ i_1 + y_2 + y_4 &\leq 30 + 29x_2 + 29x_4, \\ i_1 + y_3 + y_4 &\leq 30 + 29x_3 + 29x_4. \end{aligned}$$

3.3. Fractional Extreme Points of the LP Relaxation

Now we show that the inequalities described in §§3.1 and 3.2 cut all fractional extreme point solutions of the linear programming relaxation of LSBI. Recall that extreme point solutions are characterized by blocks $[k, l]_{\beta}^{\alpha}$ for $1 \leq k \leq l \leq n$, where $\alpha \in \{0, u_{k-1}\}$ and $\beta \in \{0, u_l\}$. Note that in an extreme point solution there can be at most one period with positive order quantity in a block.

PROPOSITION 3. *Inequalities (1), (3), and (5) cut all fractional extreme point solutions of the LP relaxation of LSBI.*

PROOF. Let (y, x, i) be an extreme point solution of the LP relaxation of LSBI. For a block type $[k, l]_0^0$, there exists one period $t \in [k, l]$ with $y_t = d_{tl} = d_{kl}$. (Note that $t = k$ if $d_k > 0$.) Therefore, the order fixed-charge variable $x_t = d_{kl}/(d_t + u_t)$. If $0 < x_t < 1$, then inequality (1),

$$y_t \leq d_{tl}x_t + i_t,$$

cuts off this point.

For a block type $[k, l]_0^{u_{k-1}}$, there exists one period $t \in [k, l]$ with $y_t = d_{kl} - u_{k-1}$. Therefore, the order fixed-charge variable $x_t = (d_{kl} - u_{k-1})/(d_t + u_t)$. If $0 < x_t < 1$, then inequality (3),

$$i_{k-1} + y_t \leq u_{k-1} + (d_{kl} - u_{k-1})x_t + i_t,$$

cuts off (y, x, i) .

For a block type $[k, l]_{u_l}^{u_{k-1}}$, there exists one period $t \in [k, l]$ with $y_t = d_{kl} + u_l - u_{k-1}$. Therefore, the order fixed-charge variable $x_t = (d_{kl} + u_l - u_{k-1})/(d_t + u_t)$. If $0 < x_t < 1$, then inequality (5),

$$i_{k-1} + y_t \leq u_{k-1} + (d_{kl} + u_l - u_{k-1})x_t,$$

cuts off (y, x, i) .

Finally, for a block type $[k, l]_{u_l}^0$, there exists one period $t \in [k, l]$ with $y_t = d_{kl} + u_l$. In this case, we must have $d_{kl} + u_l = d_t + u_t$. Therefore, the order fixed-charge variable is integral. \square

3.4. Separation

Now we discuss how to find inequality (3) violated by a given point $(y, x, i) \in \mathbb{R}_+^{3n+1}$. Let $\Delta = \sum_{t \in S} (y_t - \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t)$. For fixed k and l such that $u_{k-1} < d_{kl}$, Δ is maximized by placing $t \in [k, l]$ in S if and only if $y_t > \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t$. Because this can be done in linear time for each k and l , the observation leads to an $O(n^3)$ separation algorithm for inequality (3). Next, we improve the computational complexity of separation.

THEOREM 4. *There is an $O(n^2 \log n)$ algorithm to solve the separation problem for inequalities (3). There are $O(n \log n)$ algorithms to solve the separation problems for inequalities (1) and (5).*

PROOF. For $k \in [1, n-1]$, let $p(k) = \min\{t \in [k, l]: u_{k-1} < d_{kt}\}$ ($p(k) = l+1$ if $u_{k-1} \geq d_{kl}$). Also, let $r' = \max\{t \in [1, l]: u_t < d_{(t+1)l}\}$. Therefore, in inequality (3), x_t for $t \in S$ has coefficient $d_{kt} + u_t - u_{k-1}$ if $t \in [k, \min\{r', p(k)\}]$, $d_{kl} - u_{k-1}$ if $t \in [r' + 1, p(k)]$, and d_{tl} if $t \in [\max\{r', p(k)\} + 1, l]$. (Observe that to obtain a strong inequality, we must have $S \cap [p(k) + 1, r'] = \emptyset$ for $r' > p(k)$.)

In the rest of the discussion, we fix l and decrement k from l to 1. A set S that maximizes Δ for all $k \in [1, l]$ will be computed in $O(n \log n)$. The following observations are due to Assumption 3: (a) $p(k-1) \leq p(k)$, and (b) $d_{kl} - u_{k-1} \leq d_{(k-1)l} - u_{k-2}$.

Let $S(k)$ be a subset of $[k, l]$ maximizing Δ for k (and fixed l). Observation (b) implies that if $t \in [k, r']$ and $r' \leq p(k)$ and $t \notin S(k)$, then $t \notin S(j)$ for any $j \leq k$; also, if $t \in [r' + 1, p(k)]$ and $t \notin S(k)$, then $t \notin S(j)$ for any $j \leq k$ such that $t \leq p(j)$. However, t may be in $S(j)$ if $p(j) < t$. It is clear that if $t \in [\max\{r', p(k)\} + 1, l]$ and $t \notin S(k)$, then $t \notin S(j)$ for any $j \leq k$. Also note that $S(j) \cap [p(j) + 1, r'] = \emptyset$ for $r' > p(k)$.

Let T be the list of $t \in [k, r']$ for $r' \leq p(k)$ such that $\psi_t := y_t/x_t - (d_{kt} + u_t - u_{k-1}) > 0$, sorted by ψ_t ; T' be the list of $t \in [r' + 1, p(k)]$ such that $\varepsilon_t := y_t/x_t - (d_{kl} - u_{k-1}) > 0$, sorted by ε_t ; and, finally, T'' be the (unsorted) list of $t \in [\max\{r', p(k)\} + 1, l]$ such that $y_t - d_{tl}x_t > 0$. For $k' = k-1$, if $k' \leq r'$, then we place k' into T in $O(\log n)$ time by binary search on ψ_t and delete permanently from T all $t \leq r'$ with $\psi'_t = \psi_t - (d_{k-1} - u_{k-2} + u_{k-1}) \leq 0$. On the other hand, if $r' < k'$, then we place k' into T' in $O(\log n)$ time by binary search on ε_t . We delete from T' all $t > p(k')$ and all $t \in T'$ such that $\varepsilon'_t = \varepsilon_t - (d_{k-1} - u_{k-2} + u_{k-1}) \leq 0$ permanently. Finally, $t \in [p(k') + 1, p(k)]$ is placed in T'' permanently if and only if $y_t - d_{tl}x_t > 0$. Thus, $S(k') = T \cup T' \cup T''$.

Keeping separate sums $y(T) = \sum_{t \in T} y_t$, $x(T) = \sum_{t \in T} x_t$ and $y(T') = \sum_{t \in T'} y_t$, $x(T') = \sum_{t \in T'} x_t$ allows us to compute Δ for $S(k)$ in constant average time. Because variables are inserted into and deleted from each list at most once, all updates in Δ for the changes in the lists can be done in $O(n)$. For the elements remaining in the lists, updates can be done in constant time per iteration. For instance, let $\bar{T} \subseteq T$ be the set of elements that remain from one iteration to the next. Because the coefficients of x_i for all $i \in \bar{T}$ reduce by $d_{k-1} - u_{k-2} + u_{k-1}$, the required change for them in Δ is $-(d_{k-1} - u_{k-2} + u_{k-1})x(\bar{T})$.

There are at most n insertions to T and T' , all of which take total $O(n \log n)$. Also, there are at most n deletions from T' and insertions to T'' , all of which can be done in total $O(n)$. Because Δ can be updated in a total of $O(n)$, the complexity of the algorithm is $O(n \log n)$ for each l , giving an overall $O(n^2 \log n)$ separation algorithm for inequality (3).

From the above algorithm, it also follows that the separation for the simpler inequality (5) can be done in $O(n \log n)$ by fixing $l = n$ only. Separation for inequality (1) can be done similarly in $O(n \log n)$ by keeping the elements of T'' sorted by $y_t/x_t - d_{tl}$ and incrementing l from 1 to n . \square

4. Fixed Costs on Inventory

In this section, we expand the study to incorporate the inventory fixed-charge variables. Thus, the polyhedron studied in this section is

$$\mathcal{Q} = \text{conv} \left\{ \begin{array}{l} i_{t-1} + y_t - i_t = d_t, \quad t \in [1, n], \\ 0 \leq i_t \leq u_t z_t, \quad t \in [0, n], \\ 0 \leq y_t \leq (d_t + u_t)x_t, \quad t \in [1, n], \\ y \in \mathbb{R}^n, x \in \{0, 1\}^n, \quad i \in \mathbb{R}^{n+1}, z \in \{0, 1\}^{n+1}. \end{array} \right\}$$

4.1. Uncapacitated Inequalities

As before, we first describe inequalities that do not consider the inventory upper bounds. For $t \in [0, n-1]$, let b_t be the first time period after t with positive demand, i.e., $b_t = \min\{k \in [t+1, n]: d_k > 0\}$. Then, because d_{b_t} is either satisfied from inventory at t or from an order in a later period up to b_t , inequality

$$z_t + x_{t+1} + \dots + x_{b_t} \geq 1 \quad (7)$$

is valid for \mathcal{Q} .

For $0 \leq k \leq l \leq n$ and $S \subseteq [k, l]$, inequality

$$i_{k-1} + \sum_{t \in S} y_t \leq d_{kl} z_{k-1} + \sum_{t \in S} d_{tl} x_t + i_l, \quad (8)$$

where i_{-1} and z_{-1} are taken as 0, generalizes (7) as well as the (l, S) inequalities (1). When there are no inventory upper bounds, van Vyve and Ortega (2003) show that it suffices to add inequalities (8) to the LP relaxation of the uncapacitated lot-sizing problem with inventory fixed costs to obtain the convex hull of the feasible solutions. Inequalities (8) are not sufficient to describe \mathcal{Q} ; however, they define facets of \mathcal{Q} , as shown in the appendix.

4.2. Capacitated Inequalities

To obtain strong inequalities that use the inventory fixed costs as well as the upper bounds, we introduce the inventory fixed-charge variables into inequalities (3) and (5).

For $1 \leq k \leq l \leq n$ such that $u_{k-1} \leq d_{kl}$, let $S \subseteq [k, l]$ and $T := \{t_1, t_2, \dots, t_\tau\} \subseteq [k-1, p-1]$. For $j \in T$, let $s(j) = \min\{t \in S \cup \{l+1\}: t > j\}$. Consider the inequality

$$\begin{aligned} i_{k-1} + \sum_{t \in S} y_t + \sum_{t \in T} \gamma_t (1 - z_t) \\ \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\} x_t + i_l, \end{aligned} \quad (9)$$

where

$$\gamma_{t_j} = \begin{cases} u_{k-1} - d_{kt_j} & \text{if } j = \tau \text{ and } s(t_j) > p, \\ d_{(t_j+1)(t_{j+1})} & \text{if } j < \tau \text{ and } s(t_j) = s(t_{j+1}), \\ d_{(t_j+1)(s(t_j)-1)} & \text{if } (j < \tau \text{ and } s(t_j) < s(t_{j+1})) \\ & \text{or } (j = \tau \text{ and } s(t_j) \leq p). \end{cases} \quad (10)$$

EXAMPLE 1 (continued). Consider inequality (3) for $[k, l] = [2, 5]$ (where $p = 4$) and $S = \{2\}$:

$$i_1 + y_2 \leq 30 + 24x_2 + i_5, \quad (11)$$

which is facet defining for the restriction $z_t = 1$ for $t \in [0, 5]$. Now if $z_2 = 0$, then we observe that $i_1 + y_2 \leq 12 = d_2$. Therefore,

$$i_1 + y_2 + 18(1 - z_2) \leq 30 + 24x_2 + i_5$$

is valid for \mathcal{Q} , where $T = \{2\}$, $t_\tau = 2$, and $s(2) = l + 1 = 6 > p$, so $\gamma_2 = u_1 - d_2 = 18$.

Alternatively, for inequality (11), if $z_3 = 0$, then $i_1 + y_2 \leq 25 = d_{23}$. Therefore,

$$i_1 + y_2 + 5(1 - z_3) \leq 30 + 24x_2 + i_5, \quad (12)$$

where $\gamma_3 = u_1 - d_{23} = 5$. Now for (12), if $z_2 = 0$, then $i_1 + y_2 + 5(1 - z_3) \leq 12 + 5 = d_2 + \gamma_3$ and inequality

$$i_1 + y_2 + 13(1 - z_2) + 5(1 - z_3) \leq 12 + 5 = d_2 + \gamma_3,$$

which is equivalent to inequality (9) with $T = \{2, 3\}$, $s(2) = s(3) = 6$, and $\gamma_2 = d_3 = 13$.

This example illustrates that γ in (10) are sequence-dependent lifting coefficients for z_t , $t \in T$. Next, we give a direct proof of validity.

PROPOSITION 5. Inequality (9) is valid for \mathcal{Q} .

PROOF. Let $(y, x, i, z) \in \mathcal{Q}$. If $z_t = 1$ for all $t \in T$, validity follows from Proposition 1. Therefore, we assume that $z_t = 0$ for some $t \in T$. For a given point $(y, x, i, z) \in \mathcal{Q}$, for $1 \leq k \leq l \leq n$, $S \subseteq [k, l]$ and $T \subseteq [k-1, p-1]$, if $z_t = 0$ for some $t \in T$, then let $V = \{v_1, v_2, \dots, v_\kappa\} = \cup_{j \in S \cup \{l+1\}} (\min\{t \in T: z_t = 0, s(t) = j\})$. Without loss of generality, assume that $v_1 < v_2 < \dots < v_\kappa$; $v_0 = 0$ and $v_{\kappa+1} = n$. Thus, $S \cap [s, p-1]$ is partitioned as $(S_1, S_2, \dots, S_\kappa, S_{\kappa+1})$, where $S_j = \{t \in S: v_{j-1} < t \leq v_j\}$, and T is partitioned as $(T_1, T_2, \dots, T_\kappa, T_{\kappa+1})$, where $T_j = \{t \in T: s(v_{j-1}) \leq t < s(v_j)\}$ with $s(v_0) = k-1$ and $s(v_{\kappa+1}) = n$. Observe that for $t_j \in T$ if $s(t_j) \leq p$,

$$\sum_{t \in T \cap [t_j, s(t_j)-1]} \gamma_t \leq d_{(t_j+1)(s(t_j)-1)} \quad (13)$$

and if $s(t_j) > p$,

$$\sum_{t \in T \cap [t_j, s(t_j)-1]} \gamma_t \leq u_{k-1} - d_{kt_j}. \quad (14)$$

First, we express the left-hand side of (9) as the sum of $\kappa + 1$ terms: $i_{k-1} + \sum_{t \in S: t \leq v_1} y_t + \sum_{t \in T: t < s(v_1)} \gamma_t (1 - z_t)$, $\sum_{t \in S: v_{j-1} < t \leq v_j} y_t + \sum_{t \in T: s(v_{j-1}) \leq t < s(v_j)} \gamma_t (1 - z_t)$ for $j \in [2, \kappa]$ and $\sum_{t \in S: t > v_\kappa} y_t$. Next, we show the following relations for the first κ of these terms.

Observe from (13) that if $s(v_1) \leq p$,

$$i_{k-1} + \sum_{t \in S: t \leq v_1} y_t + \sum_{t \in T: t < s(v_1)} \gamma_t(1 - z_t) \leq d_{kv_1} + d_{(v_1+1)(s(v_1)-1)} = d_{k(s(v_1)-1)}, \tag{15}$$

because $i_{k-1} + \sum_{t \in S: t \leq v_1} y_t \leq d_{kv_1}$ when $z_{v_1} = i_{v_1} = 0$. Similarly, if $s(v_1) > p$ (in which case $\kappa = 1$), then

$$i_{k-1} + \sum_{t \in S: t \leq v_1} y_t + \sum_{t \in T: t < s(v_1)} \gamma_t(1 - z_t) \leq d_{kv_1} + d_{v_1 t_\tau} + u_{k-1} - d_{kt_\tau} = u_{k-1}. \tag{16}$$

For $v_j, j \in [2, \kappa]$ with $s(v_j) \leq p$, we have

$$\sum_{t \in S: v_{j-1} < t \leq v_j} y_t + \sum_{t \in T: s(v_{j-1}) \leq t < s(v_j)} \gamma_t(1 - z_t) \leq d_{(s(v_{j-1}))v_j} + d_{(v_j+1)(s(v_j)-1)} = d_{(s(v_{j-1}))(s(v_j)-1)}. \tag{17}$$

On the other hand, for v_κ if $\kappa \geq 2$ and $s(v_\kappa) > p$, then

$$\sum_{t \in S: v_{\kappa-1} \leq t \leq v_\kappa} y_t + \sum_{t \in T: s(v_{\kappa-1}) \leq t < s(v_\kappa)} \gamma_t(1 - z_t) \leq d_{s(v_{\kappa-1})v_\kappa} + d_{(v_\kappa+1)t_\tau} + u_{k-1} - d_{kt_\tau} = u_{k-1} - d_{k(s(v_{\kappa-1})-1)}. \tag{18}$$

To see the validity of inequality (9), we consider six cases. Let b and h be defined as in the proof of Proposition 1. First, suppose that $h \leq p$. If $s(v_\kappa) > p$, then from (15)–(18), we have

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t > v_\kappa} y_t &\leq u_{k-1} + \sum_{t \in S: t > v_\kappa} y_t \\ &\leq u_{k-1} + d_{(p+1)l} + i_l \\ &\leq u_{k-1} + d_{kl} - u_{k-1} + i_l \quad (\text{because } u_{k-1} < d_{kp}) \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l, \end{aligned}$$

where the second inequality follows, because $\sum_{t \in S: t > v_\kappa} y_t \leq d_{(p+1)l} + i_l$ if $s(v_\kappa) > p$ and the last inequality follows, because $x_t \geq 0$ for $t \in S, x_h = 1$, and for $h \leq p, d_{kl} - u_{k-1} \leq \min\{d_{kt} + u_t - u_{k-1}, d_{tl}\}$.

Similarly, if $s(v_\kappa) \leq p$, then from (15) and (17),

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t > v_\kappa} y_t &\leq d_{k(s(v_\kappa)-1)} + d_{s(v_\kappa)l} + i_l \\ &= d_{kl} + u_{k-1} - u_{k-1} + i_l \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l. \end{aligned}$$

Now suppose that $h > p$. If $s(v_\kappa) > p$ and $b > v_\kappa$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_\kappa < t \leq b} y_t + \sum_{t \in S: t \geq h} y_t &\leq u_{k-1} + d_{s(v_\kappa)b} + u_b + d_{hl} + i_l \\ &= u_{k-1} + d_{kb} + u_b - u_{k-1} + d_{hl} + i_l + (u_{k-1} - d_{k(s(v_\kappa)-1)}) \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l, \end{aligned}$$

where the last inequality follows because $u_{k-1} < d_{k(s(v_\kappa)-1)}$ for $s(v_\kappa) > p$. If $s(v_\kappa) \leq p$ and $b > v_\kappa$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_\kappa < t \leq b} y_t + \sum_{t \in S: t \geq h} y_t &\leq d_{k(s(v_\kappa)-1)} + d_{s(v_\kappa)b} + u_b + d_{hl} + i_l + u_{k-1} - u_{k-1} \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l. \end{aligned}$$

If $s(v_\kappa) > p$ and $b \leq v_\kappa$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t \geq h} y_t &\leq u_{k-1} + d_{hl} + i_l \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l. \end{aligned}$$

Finally, if $s(v_\kappa) \leq p$ and $b \leq v_\kappa$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t > v_\kappa} y_t &\leq d_{k(s(v_\kappa)-1)} + d_{hl} + i_l \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{tl}\}x_t + i_l. \quad \square \end{aligned}$$

In the next class of inequalities, we introduce inventory fixed-charge variables into inequality (5). For $1 \leq k \leq l \leq n$, let $S \subseteq [k, l]$ and $T := \{t_1, t_2, \dots, t_\tau\} \subseteq [k-1, p-1]$, where $p = \min\{t \in [k, l]: u_{k-1} < d_{kt}\}$. For $t \in T$, let $s(j) = \min\{t \in S \cup \{l+1\}: t > j\}$. Then, consider the inequality

$$i_{k-1} + \sum_{t \in S} y_t + \sum_{t \in T} \gamma_t(1 - z_t) \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t, \tag{19}$$

where γ_t is defined as in (10).

PROPOSITION 6. Inequality (19) is valid for \mathcal{Q} .

PROOF. For $(y, x, i, z) \in \mathcal{Q}$ if $x_t = 0$ for all $t \in S$, inequality follows from $i_{k-1} + \sum_{t \in T} \gamma_t(1 - z_t) \leq u_{k-1}$ due to (15)–(18). Otherwise, let $\omega = \max\{t \in S: x_t = 1\}$. Using the definitions in the proofs of Propositions 1 and 5, if $s(v_\kappa) > p$ and $\omega > v_\kappa$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_\kappa < t \leq \omega} y_t &\leq u_{k-1} + d_{s(v_\kappa)\omega} + u_\omega + (d_{kp} - u_{k-1}) \\ &\leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t. \end{aligned}$$

If $s(v_k) \leq p$ and $\omega > v_k$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_k} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_k < t \leq \omega} y_t \\ \leq d_{k(s(v_k)-1)} + d_{s(v_k)\omega} + u_\omega + u_{k-1} - u_{k-1} \\ \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t. \end{aligned}$$

Finally, if $\omega \leq v_k$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S: t \leq v_k} y_t + \sum_{t \in T} \gamma_t(1 - z_t) \\ \leq u_{k-1} \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t. \quad \square \end{aligned}$$

In the appendix we study the strength of inequalities (9) and (19) with respect to \mathcal{Q} .

EXAMPLE 1 (continued). All facet-defining inequalities (19) of \mathcal{Q} for $[k, l] = [2, 5]$ with $|S| \leq 1$ are

$$\begin{aligned} i_1 + 12(1 - z_1) + 13(1 - z_2) + 5(1 - z_3) &\leq 30, \\ i_1 + 12(1 - z_1) + 18(1 - z_2) &\leq 30, \\ i_1 + 25(1 - z_1) + 5(1 - z_3) &\leq 30, \\ i_1 + 30(1 - z_1) &\leq 30, \\ i_1 + 18(1 - z_2) + y_2 &\leq 30 + 29x_2, \\ i_1 + 13(1 - z_2) + 5(1 - z_3) + y_2 &\leq 30 + 29x_2, \\ i_1 + 12(1 - z_1) + 5(1 - z_3) + y_3 &\leq 30 + 29x_3, \\ i_1 + 25(1 - z_1) + y_4 &\leq 30 + 29x_4, \\ i_1 + 12(1 - z_1) + 13(1 - z_2) + y_4 &\leq 30 + 29x_4. \end{aligned}$$

Observe that these inequalities are extensions of inequalities (5) with inventory fixed-charge variables. The first four inequalities are extensions of inequality $i_1 \leq 30$, the next two are extensions of $i_1 + y_2 \leq 30 + 29x_2$, and so on.

On the other hand, all facet-defining inequalities (9) of \mathcal{Q} for $[k, l] = [2, 5]$ with $|S| = 1$ are

$$\begin{aligned} i_1 + 18(1 - z_2) + y_2 &\leq 30 + 24x_2 + i_5, \\ i_1 + 13(1 - z_2) + 5(1 - z_3) + y_2 &\leq 30 + 24x_2 + i_5, \\ i_1 + 12(1 - z_1) + 5(1 - z_3) + y_3 &\leq 30 + 24x_3 + i_5, \\ i_1 + 25(1 - z_1) + y_4 &\leq 30 + 24x_4 + i_5, \\ i_1 + 12(1 - z_1) + 13(1 - z_2) + y_4 &\leq 30 + 24x_4 + i_5. \end{aligned}$$

Because the extreme point solutions of the LP relaxation have the same block structure with or without inventory fixed costs, it follows from Proposition 3 and Assumption 3 that inequalities (8), (9), and (19), which are generalizations of inequalities (1), (3), and (5), cut off all fractional extreme points of the LP relaxation of LSBI.

4.3. Separation

In this section, we study the separation problems for inequalities (9) and (19) and show that they can be solved in polynomial time.

THEOREM 7. There is an $O(n^4)$ algorithm to solve the separation problem for inequalities (9) and an $O(n^3)$ algorithm to solve the separation problem for inequalities (19).

PROOF. Rewriting inequality (9) as

$$\begin{aligned} \sum_{t \in S} y_t + \sum_{t \in T} \gamma_t(1 - z_t) \\ - \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{kl} - u_{k-1}, d_{lt}\}x_t \\ \leq u_{k-1} - i_{k-1} + i_l, \end{aligned} \quad (20)$$

for each k and l such that $1 \leq k \leq l \leq n$ and $u_{k-1} < d_{kl}$, we wish to determine sets $S \subseteq [k, l]$ and $T \subseteq [k-1, p-1]$ such that the left-hand side of (20) is maximized for a given point (y, x, i, z) . We formulate this problem as a longest-path problem on a directed acyclic network. The arcs on a longest path determine the sets S and T .

Consider a directed graph $G = (V, A)$ with a single source $(k-1) \in V$ and a single sink $(l+1) \in V$. Let $t \in V$ for $t \in [k, l] \setminus [p+1, r']$, where $r' = \max\{t \in [k, l]: u_t < d_{(t+1)l}\}$ ($r' = k-1$ if $u_t \geq d_{(t+1)l}$ for all $t \in [k, p]$).

There is an arc $(t, j)^a \in A$, $a \in \{T, S\}$, for each $t \in V$ and $j \in V$ such that $k-1 \leq t < j \leq p-1$ so that if the path includes arc $(t, j)^a$, then j is included in both S and T if $a = S$, j is included in T if $a = T$ ($k-1$ is assumed to be in T because $\{k-1\} \cup (S \cap [k, p-1]) \subseteq T \subseteq [k-1, p-1]$ for strong inequalities). Also, for $k \leq t \leq p-1 < j \leq l+1$ with $t, j \in V$, arc $(t, j)^S$ is in A and if the path includes this arc, then $j \in S$. For $p \leq t < j \leq l+1$ with $t, j \in V$, arc $(t, j)^S$ is in A and if the path includes this arc, then $j \in S$. Finally, let $((k-1), p)^S$ be an arc in A ; if the path includes this arc, then $p \in S$. (Note that there is no arc $(k, j)^S$ for $j > p$, because we must have $S \cap [k, p] \neq \emptyset$ from Condition 6 of Proposition 13.)

Now we assign lengths to the arcs. Let the length of arc $(t, j)^T$ for $k-1 \leq t < j \leq p-1$ be

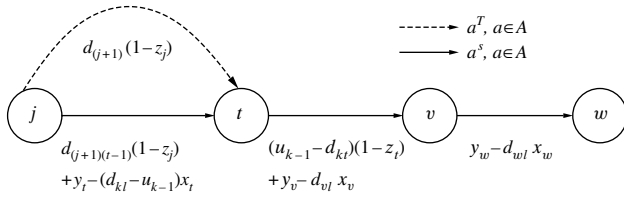
$$c_{(t,j)^T} = d_{(t+1)j}(1 - z_t).$$

Also, let the length of arc $(t, j)^S$ be

$$c_{(t,j)^S} = \begin{cases} d_{(t+1)(j-1)}(1 - z_t) + y_j - \min\{d_{kt} + u_t - u_{k-1}, \\ d_{kl} - u_{k-1}\}x_j & \text{if } k-1 \leq t < j \leq p, \\ (u_{k-1} - d_{kt})(1 - z_t) + y_j - d_{jt}x_j & \text{if } k \leq t < p < j \leq l+1, \\ y_j - d_{jt}x_j & \text{if } p \leq t < j \leq l+1, \end{cases}$$

where we use $y_{l+1} - d_{(l+1)l}x_{l+1} = 0$, for simplicity of notation. It is easy to see that the total length of a path from

Figure 4. Network for separation for inequality (9) ($k \leq j < t < p < v < w \leq l$).



vertex $(k - 1)$ to vertex $(l + 1)$ equals the left-hand side of the corresponding inequality (20).

Different arc types and costs for $k \leq j < t < p < v < w \leq l$ are illustrated in Figure 4. For simplicity, not all the arcs in the subgraph induced by j, t, v, w are shown in the figure.

Note that G is a directed acyclic graph with $O(n)$ vertices and $O(n^2)$ arcs. Therefore, the longest path in G can be found in $O(n^2)$ time (see Ahuja et al. 1993). Because we solve a longest-path problem for each k and l such that $u_{k-1} < d_{kt}$, the overall complexity of the separation algorithm for inequalities (9) is $O(n^4)$. The graph G can be preprocessed to remove low-length parallel arcs between two nodes, but this does not change the complexity of the algorithm.

It follows that separation for inequalities (19) is done in $O(n^3)$ by letting $l = n$. \square

Separation for inequalities (8) is done in $O(n \log n)$ similar to separation for inequalities (1).

4.4. A Special Case: Wagner-Whitin Nonspeculative Costs

In this section, we consider the special case of LSBI, in which the objective function satisfies the *Wagner-Whitin nonspeculative property*, i.e., $p_t + h_t \geq p_{t+1}$ for all $t \in [0, n]$, where $p_0 = p_{n+1} = 0$. For this case, Pochet and Wolsey (1994) give an LP formulation for the lot-sizing problem with inventory bounds, but no inventory fixed costs. On the other hand, van Vyve and Ortega (2003) give an LP formulation for the lot sizing with inventory fixed costs, but no bounds. In this section, we generalize these results to the lot-sizing problem with inventory bounds and fixed costs by observing that the intersection of the previously given LP formulations is an integral polyhedron.

Unlike in the general cost case, if the Wagner-Whitin nonspeculative property holds, then there is an optimal *inventory-minimal solution*; that is, the demand in each period $t \in [1, n]$ is satisfied either from inventory or by ordering in that period, but not both, because otherwise, because the order quantity is unbounded, we can increase y_t until $i_{t-1} = 0$ without increasing the objective value.

PROPOSITION 8. *If $p_t + h_t \geq p_{t+1}$ for all $t \in [0, n]$, where $p_0 = p_{n+1} = 0$, then $\min\{py + fx + hi + gz : (y, x, i, z) \in \mathcal{Q}\}$ has an optimal solution that satisfies $i_{t-1}y_t = 0$ for all $t \in [1, n]$.*

Thus, if the costs satisfy the Wagner-Whitin nonspeculative property, then there exists an optimal solution that consists of blocks of type $[k, l]_0^0$ only, which implies that, in this case, LSBI can be solved as the uncapacitated lot-sizing problem by simply ignoring solutions with blocks $[k, l]_0^0$ that do not satisfy inventory upper bounds.

Next, we describe a formulation of LSBI over the inventory-minimal solutions. Note that (y, x, i, z) is an inventory-minimal solution if and only if for all $t \in [1, n]$ we have $i_{t-1} = d_{tj}$ for some $j \in [t, n]$. Let ρ_{tk} be a binary variable such that $\rho_{tk} = 1$ if $i_t \geq d_{(t+1)k} > 0$ and $\rho_{tk} = 0$ otherwise for $k \in [t + 1, n]$. Then, inventory-minimal solutions can be modeled as $i_t = \sum_{k \in [t+1, n]} d_k \rho_{tk}$ and $\rho_{tk} = (1 - \sum_{j \in [t+1, k]} x_j)^+$ for $k \in [b_t, n]$, where b_t is the first period after t with positive demand, i.e., $b_t = \min\{k \in [t + 1, n] : d_k > 0\}$. Observe that the second set of equalities implies $\rho_{tk} \geq \rho_{t(k+1)} \geq \dots \geq \rho_{tn}$; hence, $i_t = d_{(t+1)j}$ for some $j \in [t + 1, n]$ (see Pochet and Wolsey 1994).

Furthermore, for an inventory-minimal solution, $i_t \leq u_t z_t$ if and only if (24) $z_t \geq \rho_{tb_t}$ and (23) $\sum_{j \in [t+1, l_t]} x_j \geq 1$, where l_t is the first time period after t in which demand cannot be completely satisfied by i_t , i.e., $l_t = \min\{k \in [t + 1, n] : u_t < d_{(t+1)k}\}$. To see this, suppose that $i_t \leq u_t z_t$ holds. Then, $d_{(t+1)(l_t-1)} \leq u_t < d_{(t+1)l_t}$, and we have (23). Also, $z_t = 0$ implies $i_t = 0$ and from $i_t = \sum_{k \in [t+1, n]} d_k \rho_{tk}$ (inventory minimality), it follows that $\rho_{tk} = 0$ for all $k \in [t + 1, n]$. Thus, we have (24) as well. For the other direction, suppose (23)–(24) hold. From (23), $x_k = 1$ for some $k \in [t + 1, l_t]$. Then, for an inventory-minimal solution, from $\rho_{tk} = (1 - \sum_{j \in [t+1, k]} x_j)^+$, we have $\rho_{tk} = \dots = \rho_{tn} = 0$ and $i_t \leq d_{(t+1)(k-1)} \leq u_t$. From (24), if $z_t = 0$, then $\rho_{tb_t} = 0$, which, by $\rho_{tk} = (1 - \sum_{j=t+1}^k x_j)^+$ implies that $x_j = 1$ for some $j \in [t + 1, b_t]$. Then, again by inventory minimality, $\rho_{tb_t} = \dots = \rho_{tn} = 0$ and $i_t = 0$ as $d_{(t+1)b_t} - 1 = 0$.

After eliminating the order variables y_t by substituting $d_t + i_t - i_{t-1}$, the inventory costs become $h'_t = p_t + h_t - p_{t+1}$ and the objective contains the constant term $K = \sum_{t \in [1, n]} p_t d_t$. Because $h' \geq 0$ due to the Wagner-Whitin nonspeculative costs, the problem can be stated as

$$\min \sum_{t \in [1, n]} f_t x_t + \sum_{t \in [0, n]} (g_t z_t + h'_t i_t) + K$$

$$i_t \geq \sum_{k \in [t+1, n]} d_k \rho_{tk}, \quad t \in [0, n-1], \tag{21}$$

$$\rho_{tk} + \sum_{j \in [t+1, k]} x_j \geq 1, \quad t \in [0, n-1], k \in [b_t, n], \tag{22}$$

$$\sum_{j \in [t+1, l_t]} x_j \geq 1, \quad t \in [0, n-1], \tag{23}$$

$$z_t - \rho_{tb_t} \geq 0, \quad t \in [0, n-1], \tag{24}$$

$$\rho_{tk} \geq 0, \quad t \in [0, n-1], k \in [t+1, n], \tag{25}$$

$$0 \leq x_t \leq 1, \quad t \in [1, n], \tag{26}$$

$$0 \leq z_t \leq 1, \quad t \in [0, n], \tag{27}$$

$$i \in \mathbb{R}_+^{n+1}, \quad x \in \{0, 1\}^n, \quad z \in \{0, 1\}^{n+1}, \tag{28}$$

$$\rho \in \{0, 1\}^{\binom{n}{2}}.$$

Pochet and Wolsey (1994) show that the constraint matrix for (22)–(23) and (25)–(27) is totally unimodular. Because each z_t appears once in (24), the following statement holds.

PROPOSITION 9. *Let $w = (\rho, x, z)$ and let $Aw \geq b$ represent constraints (22)–(27). The constraint matrix A is totally unimodular.*

Because the right-hand side of constraints (22)–(27) is integral and each inventory variable i_t appears only once in constraint (21), the extreme points of

$$\mathcal{Q}_{WW} = \{(x, i, z, \rho) : (x, i, z, \rho) \text{ satisfies (21)–(27)}\},$$

are the inventory-minimal solutions of LSBI.

COROLLARY 10. *The linear program $\min\{fx + hi + gz + K : (x, i, z, \rho) \in \mathcal{Q}_{WW}\}$ is an extended formulation of LSBI with Wagner-Whitin nonspeculative costs.*

Projecting out the auxiliary variables ρ in \mathcal{Q}_{WW} and reintroducing order variables y , we obtain a linear programming formulation of the problem with the original variables.

THEOREM 11. *LSBI with Wagner-Whitin nonspeculative costs can be formulated as the following linear program:*

$$\min\{py + fx + hi + gz : (y, x, i, z) \text{ satisfies (29)–(35)}\},$$

where

$$\sum_{j \in [t, k]} (y_j - d_{jk}x_j) \leq i_k, \quad 1 \leq t \leq k \leq n, \quad (29)$$

$$\sum_{j \in [t, l_{t-1}]} x_j \geq 1, \quad t \in [1, n], \quad (30)$$

$$z_{t-1} + \sum_{j \in [t, b_{t-1}]} x_j \geq 1, \quad t \in [1, n], \quad (31)$$

$$i_{t-1} + y_t = d_t + i_t, \quad t \in [1, n], \quad (32)$$

$$0 \leq y_t, \quad t \in [1, n], \quad (33)$$

$$0 \leq x_t \leq 1, \quad t \in [1, n], \quad (34)$$

$$0 \leq z_t \leq 1, \quad t \in [0, n]. \quad (35)$$

Note that inequality (29) is a special case of (l, S) inequalities with $S = [k, l]$, inequality (30) is a special case of inequality (3) with $S = [k, l_k]$, and inequality (31) is a special case of inequality (8) with $S = [k, b_{k-1}]$.

5. Computational Results

To test the effectiveness of the inequalities described in §§3 and 4 in solving LSBI, we implement a branch-and-cut algorithm that incorporates these inequalities and we perform computational experiments. All computations are done on a 2 GHz Pentium 4/Linux workstation with 1 GB main memory.

The data used in the experiments has the following properties: Demands are generated from integer uniform distribution between 0 and 30. Inventory upper bounds are generated from integer uniform distribution between 30 and $30(c + 1)$, where $c \geq 0$ is a parameter to determine tightness of capacity. Order costs are generated from integer uniform distribution between 4 and 24, and holding cost is equal to 1 for every period. Let f be the ratio of order fixed cost to variable inventory cost and g be the ratio of order fixed cost to inventory fixed cost. To observe the effect of varying capacity and cost parameters on the computations, we let $c \in \{2, 5, 10, 20\}$, $f \in \{1,000, 2,000, 5,000\}$, and $g \in \{2, 5, 10\}$, and generate five random instances for each combination.

The first set of experiments is on solving LSBI with linear inventory costs. The problem instances are solved with the MIP solver of CPLEX¹ Version 8.1 Beta using first only the uncapacitated inequality (1) (Unc) and then using inequalities (1), (3), and (5) (Cap) as cutting planes in the branch-and-cut tree. Given a fractional point, we find violated inequalities (1), (3), and (5) using the separation algorithms discussed in §3.4. CPLEX cuts are disabled in these experiments to isolate the impact due to the inequalities discussed in this paper. However, to see how CPLEX cuts would perform, we also solve the same instances with the default settings of CPLEX (Def) without adding any user cuts.

A summary of these experiments for instances with 120 and 180 time periods is reported in Tables 1 and 2. In the second column of the tables we report the average integrality gap, which is $100 \times (\text{zub} - \text{zinit})/\text{zub}$, where zinit is the objective value of the initial LP relaxation and zub is the objective value of the best integer solution. In the third column, we compare the average percentage improvement of the integrality gap at the root node (% gapimp), which is $100 \times (\text{zroot} - \text{zinit})/(\text{zub} - \text{zinit})$, where zroot is the objective value of the LP at the root node after the cuts are added. Columns cuts and nodes compare the average number of cuts added, and the average number of branch-and-cut nodes explored, respectively. In the last column, we report the average CPU time elapsed (in seconds) if the problem is solved within a one-hour time limit. Otherwise, we also report, in parentheses, the average percentage gap between the best lower bound and the best integer solution found in the search tree (endgap) and the number of instances that have positive end gap (unslv). Except for percentage gaps, the entries in the tables are rounded to a nearest integer.

In Tables 1 and 2, we observe that the initial integrality gap increases with the length of the planning horizon and the ratio of order fixed cost to linear inventory cost f . The capacitated cuts close almost all of the integrality gap for most instances and reduce the computational effort dramatically. Indeed, many instances are solved without the need for branching, and the rest require only a few branches to prove optimality. While many of the instances cannot be

Table 1. LSBI with linear inventory holding costs, $n = 120$.

f	c	gap	% gapimp			cuts			nodes			time (endgap:unslv)		
			Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap
1,000	2	17.8	73.5	22.1	95.2	195	42	482	93,655	1,220,825	7	360	3,574 (6.9:2)	2
	5	17.1	77.7	34.5	96.9	131	60	504	9,109	579,048	2	31	2,285 (1.7:2)	3
	10	17.5	81.1	51.2	99.6	97	81	529	13,495	65,678	1	37	335	2
	20	17.8	83.5	67.4	99.7	72	111	525	1,666	2,793	1	5	17	2
2,000	2	19.6	60.7	14.3	95.9	198	35	560	700,111	1,388,269	3	2,207 (1.6:3)	3,614 (10.5:5)	2
	5	18.6	54.2	21.7	95.1	150	49	492	701,982	1,151,008	6	1,884 (0.9:2)	3,613 (5.3:5)	3
	10	18.7	67.0	36.2	99.6	108	73	533	308,372	675,722	1	867	2,637 (2.4:3)	3
	20	17.2	80.7	53.8	99.5	80	120	569	447	16,085	1	2	87	2
5,000	2	22.5	44.9	6.2	96.9	192	28	528	1,182,451	1,554,172	2	2,936 (6.5:4)	3,614 (15.7:5)	2
	5	21.0	60.3	10.7	94.8	194	35	493	501,072	1,225,022	3	1,218 (1.9:1)	3,137 (9.3:4)	3
	10	20.9	53.2	19.3	100.0	384	58	591	666,193	875,933	0	2,184 (1.7:3)	2,938 (5.6:3)	3
	20	18.0	68.5	29.9	100.0	189	88	667	31,493	166,887	0	115	924 (0.9:1)	2
Average		18.9	67.1	28.5	97.8	166	65	539	350,837	743,454	2	988 (1.1:1)	2,231 (4.9:3)	2

solved to optimality in an hour of CPU time with default CPLEX and with uncapacitated lot-sizing inequalities (1), when the capacitated cuts (3) and (5) are added, all of them are solved within a few seconds. Note that uncapacitated lot-sizing inequalities (1) results in small gap improvement at the root node and have high endgap on average. As expected, the root gap improvement with uncapacitated inequalities increases with maximum capacity ratio c .

The second set of computational experiments is on solving LSBI with linear and fixed inventory costs. In these experiments we test the marginal impact of the inequalities with fixed-charge variables (8), (9), and (19) over inequalities that do not make use of the fixed-charge variables (1), (3), and (5) as well as the marginal impact of inequalities that exploit the inventory bounds and fixed charges (9) and (19) over inequalities that only use inventory fixed charges (8). Violated inequalities (8), (9), and (19) are found as described in §4.3. A summary of these experiments for instances with 120 and 180 time periods is reported in Tables 3 and 4. The columns labelled as Def

are for default CPLEX, Unc are for runs with uncapacitated inequalities (8), Lin are for runs with inequalities (1), (3), and (5), and, finally, (Cap) are for runs with inequalities (8), (9), and (19). For these instances we set the order fixed cost to inventory fixed cost ratio g equal to 10. The initial integrality gap of the LP relaxation of LSBI with inventory fixed costs is larger than for the case with linear inventory costs only. We observe that a significant effort is spent in strengthening the LP relaxations with the cutting planes; the numbers of cuts added are in the thousands.

A comparison of column Unc and Lin with Cap show that inequalities that ignore either the capacities or the inventory fixed-charge variables are not sufficient to solve the problem efficiently. We see that the addition of inequalities (9) and (19) close almost all of the integrality gap consistently for varying order fixed-cost inventory ratios and capacities. Consequently, all of the instances that could not be solved within an hour of CPU time without inequalities (9) and (19), are solved in a few minutes when they are added. These experiments clearly demonstrate the positive impact

Table 2. LSBI with linear inventory costs, $n = 180$.

f	c	gap	% gapimp			cuts			nodes			time (endgap:unslv)		
			Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap
1,000	2	17.5	73.9	17.6	99.1	271	74	879	358,306	701,225	4	2,404 (0.7:2)	3,608 (14.2:5)	9
	5	17.1	78.7	30.3	99.7	201	85	852	140,510	629,417	2	854 (0.3:1)	3,605 (8.9:5)	10
	10	17.4	82.5	47.5	99.3	182	114	818	28,232	457,648	2	181	3,606 (4.2:5)	9
	20	18.5	77.7	67.6	99.6	128	152	839	106,858	112,079	1	559	1,400	12
2,000	2	19.1	55.1	10.0	99.8	230	58	972	620,669	730,623	1	2,897 (6.4:4)	3,612 (20.6:5)	9
	5	18.2	68.7	18.7	99.7	186	72	821	397,850	705,143	1	1,527 (2.7:2)	3,614 (13.8:5)	11
	10	18.8	64.8	32.1	99.7	190	100	881	402,263	469,990	1	1,859 (1.6:2)	3,612 (8.4:5)	11
	20	19.0	70.0	51.0	99.9	169	158	738	175,411	314,706	0	1,043	3,589 (2.5:4)	13
5,000	2	22.2	44.4	5.4	100.0	237	46	956	818,413	787,667	1	3,612 (9.6:5)	3,613 (24.9:5)	9
	5	20.4	56.6	9.6	100.0	269	53	956	647,729	747,869	0	2,656 (4.5:3)	3,614 (17.8:5)	12
	10	21.2	56.4	19.7	100.0	195	75	1,053	694,361	660,580	0	2,890 (5.1:4)	3,613 (12.4:5)	10
	20	19.7	52.8	22.0	100.0	347	103	1,016	551,326	537,426	0	2,901 (2.6:4)	3,611 (8.3:5)	14
Average		19.1	65.1	27.6	99.7	217	91	898	411,827	57,119	1	1,949 (2.8:2)	3,425 (11.3:5)	11

Table 3. LSBI with fixed and linear inventory costs, $n = 120$ and $g = 10$.

f	c	gap	% gapimp				cuts				nodes				time (endgap:unslv)			
			Def	Unc	Lin	Cap	Def	Unc	Lin	Cap	Def	Unc	Lin	Cap	Def	Unc	Lin	Cap
1,000	2	24.5	82.8	49.6	67.4	99.5	373	1,210	529	2,395	116,844	66,587	758,430	1	671	3,615 (9.4:5)	3,611 (3.0:5)	51
	5	26.0	88.8	64.0	65.0	99.8	325	1,334	590	2,376	55,381	49,587	652,147	1	294	3,617 (4.8:5)	3,610 (2.9:5)	57
	10	28.1	88.1	76.6	64.1	99.9	288	1,486	689	2,531	192,749	26,742	548,199	1	731 (0.4:1)	2,693 (1.5:2)	3,610 (3.7:5)	47
	20	30.6	90.1	87.2	62.6	99.9	265	1,686	721	2,030	25,848	7,231	460,097	0	139	905 (0.2:1)	3,610 (4.8:5)	52
2,000	2	27.6	65.2	44.1	65.6	100.0	391	1,276	549	2,108	827,225	65,867	753,632	1	3,612 (5.9:5)	3,614 (14.8:5)	3,612 (4.2:5)	54
	5	28.9	81.5	62.0	61.7	99.6	345	1,327	543	2,438	447,568	50,818	675,077	4	2,206 (0.4:2)	3,613 (6.9:5)	3,613 (4.8:5)	60
	10	31.0	82.6	72.9	60.9	100.0	322	1,586	748	2,320	404,856	35,623	475,499	0	1,455 (1.4:2)	2,921 (3.7:4)	3,612 (5.9:5)	56
	20	33.1	85.8	85.8	60.7	99.9	305	1,812	921	2,333	210,006	9,894	328,056	1	961 (0.2:1)	1,343	3,610 (6.4:5)	58
5,000	2	30.8	65.4	42.9	64.0	100.0	433	1,345	512	2,552	900,144	62,652	789,896	1	3,610 (6.8:5)	3,613 (16.7:5)	3,613 (6.0:5)	63
	5	31.4	63.9	57.2	59.3	99.9	356	1,343	549	2,755	919,685	58,119	820,783	0	3,612 (6.5:5)	3,613 (10.5:5)	3,613 (6.9:5)	59
	10	34.0	72.3	65.5	57.0	100.0	417	1,484	641	2,256	764,752	46,794	583,127	0	2,891 (4.9:3)	3,613 (7.9:5)	3,611 (8.6:5)	60
	20	35.1	78.1	80.1	56.4	100.0	525	1,838	818	2,113	560,411	21,298	337,941	0	2,893 (2.0:4)	2,680 (1.9:3)	3,611 (8.5:5)	57
Average		30.1	78.7	65.7	62.1	99.9	362	1,477	651	2,351	452,122	41,768	598,574	1	1,923 (2.4:2)	2,978 (6.5:4)	3,611 (5.5:5)	56

of inequalities (9) and (19) over uncapacitated inequalities (8) that use fixed charge variables, as well as inequalities (8), (9), and (19) over inequalities (1), (3), and (5) that do not incorporate fixed-charge variables.

Table 5 summarizes the effect of changing the order fixed cost to inventory fixed-cost ratio g . We observe that the smaller the ratio is, the bigger the initial integrality gap. However, although the initial gap is higher, CPLEX cuts (especially flow cover cuts) lead to a better gap improvement at the root node for the instances with small g . It is remarkable that even with 97%–98% gap improvement at the root node, thousands of branch-and-cut nodes are needed to obtain provably optimal solutions. However, for the case $g = 2$ and $c = 10$, default CPLEX finishes computations more quickly even when many nodes are explored. We also note that the performance of our branch-and-cut algorithm is not affected significantly by the tightness of the inventory upper bounds, whereas the performance of the alternative branch-and-cut algorithms degrades with lower c .

Our experiments with problems that satisfy the Wagner-Whitin nonspeculative cost structure show that the compact

linear program in Theorem 11 is solved faster than the cutting-plane algorithm that starts with the standard formulation (LSBI). Finally, we compare the performance of a branch-and-bound algorithm that solves the strengthened formulation given by (29)–(35) (Compact) with the performance of the branch-and-cut algorithm (Cap) that uses inequalities (8), (9), and (19) on instances that do not satisfy the Wagner-Whitin nonspeculative cost structure. For this experiment, $n = 180$, $g = 10$, and order variable costs are drawn from discrete uniform distribution between 4 and 104. We increase the variance in the order variable costs so that they are not approximately nonspeculative. Recall that the strengthened formulation contains a subset of the inequalities generated by the branch-and-cut algorithm. In Table 6 we report the initial integrality gap of the test problems (gap), the number of branch-and-cut nodes explored, and the time to solve the problems. For the branch-and-cut algorithm, under columns rgap and cuts, we report the gap at the root node when all violated inequalities (8), (9), and (19) are added and the number of cuts added, respectively. We observe that the initial integrality gap at the root node for the compact formulation

Table 4. LSBI with fixed and linear inventory costs, $n = 180$ and $g = 10$.

f	c	gap	% gapimp				cuts				nodes				time (endgap:unslv)			
			Def	Unc	Lin	Cap	Def	Unc	Lin	Cap	Def	Unc	Lin	Cap	Def	Unc	Lin	Cap
1,000	2	24.7	84.5	45.6	70.8	99.4	592	2,598	747	6,388	279,834	22,117	505,488	15	2,909 (1.1:3)	3,629 (13.9:5)	3,613 (4.5:5)	418
	5	26.1	88.9	56.8	66.2	99.9	498	2,778	885	7,198	83,519	20,199	410,016	1	753	3,629 (10.6:5)	3,614 (5.4:5)	391
	10	28.7	90.1	69.4	64.3	99.6	467	3,074	940	7,777	93,670	16,162	410,060	3	926 (0.0:1)	3,627 (7.0:5)	3,612 (6.6:5)	359
	20	31.4	86.0	84.4	63.6	99.9	413	3,546	1,079	6,454	345,818	9,770	323,822	1	2,780 (0.6:2)	3,624 (1.9:5)	3,612 (7.6:5)	410
2,000	2	27.7	67.8	41.0	68.3	99.7	518	2,851	680	4,735	430,593	21,937	541,781	4	2,965 (7.7:4)	3,630 (18.8:5)	3,613 (6.1:5)	420
	5	29.1	76.5	54.6	63.9	99.9	476	2,929	819	7,081	368,271	20,142	405,146	1	2,424 (4.4:3)	3,628 (13.3:5)	3,610 (7.5:5)	433
	10	31.9	75.4	67.8	61.4	99.9	441	3,365	956	5,653	466,267	12,844	336,865	2	3,117 (4.7:4)	3,631 (8.5:5)	3,611 (8.7:5)	409
	20	34.8	74.3	76.6	60.1	99.9	375	3,871	1,109	4,699	609,782	9,226	209,676	2	2,916 (4.9:4)	3,632 (5.8:5)	3,612 (10.1:5)	468
5,000	2	31.0	64.6	41.3	66.7	99.9	536	2,891	732	5,134	410,827	20,719	481,684	2	2,930 (10.5:4)	3,629 (20.3:5)	3,614 (7.6:5)	432
	5	31.8	71.1	54.2	61.5	100.0	498	2,914	824	3,983	459,227	21,050	419,505	0	2,923 (7.3:4)	3,629 (14.7:5)	3,612 (9.1:5)	389
	10	34.5	71.5	61.9	57.9	100.0	386	3,242	863	5,768	562,321	17,503	385,868	1	2,898 (7.2:4)	3,631 (12.4:5)	3,613 (10.6:5)	378
	20	36.9	71.2	72.3	56.7	100.0	379	3,755	1,263	5,720	595,220	13,545	219,399	0	2,930 (7.2:4)	3,655 (8.4:5)	3,613 (12.9:5)	437
Average		30.7	76.8	60.5	63.5	99.9	465	3,151	908	5,882	392,115	17,101	387,443	3	2,539 (4.6:3)	3,631 (11.3:5)	3,612 (8.1:5)	412

Table 5. LSBI with fixed and linear inventory costs, $n = 180$ and $f = 5000$.

g	c	gap	% gap _{imp}			cuts			nodes			time (endgap:unslv)		
			Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap
10	2	30.5	59.3	40.5	100.0	491	2,998	5,636	565,010	21,855	2	3,610 (11.4:5)	3,631 (20.8:5)	501
	5	32.3	78.7	49.0	100.0	462	2,904	10,692	457,550	20,259	1	2,899 (4.5:4)	3,630 (17.9:5)	490
	10	35.1	65.9	60.5	100.0	433	3,329	4,974	685,967	18,336	1	3,610 (9.9:5)	3,635 (13.3:5)	457
	20	39.0	70.5	71.9	100.0	542	4,292	7,763	649,701	11,277	1	3,611 (8.5:5)	3,660 (9.2:5)	484
5	2	40.3	70.2	62.2	100.0	539	4,004	4,762	554,394	15,671	2	3,611 (10.7:5)	3,630 (15.8:5)	471
	5	43.7	75.5	71.7	100.0	441	3,899	6,373	664,511	16,873	1	3,612 (8.3:5)	3,629 (12.0:5)	492
	10	47.8	73.9	77.8	100.0	434	4,181	7,647	719,258	14,012	2	3,611 (9.4:5)	3,654 (9.7:5)	545
	20	52.6	75.8	86.0	99.9	430	4,771	6,027	665,906	9,885	1	3,599 (8.4:4)	3,637 (5.8:5)	631
2	2	59.9	95.1	85.0	100.0	572	4,817	6,544	266,314	11,101	2	2,329 (0.8:3)	3,631 (8.5:5)	562
	5	64.5	95.8	91.9	100.0	485	4,835	5,416	174,766	10,936	1	1,128 (0.6:1)	3,629 (4.5:5)	572
	10	68.5	98.0	92.7	100.0	452	4,934	5,568	9,169	12,206	1	82	3,631 (4.3:5)	602
	20	72.5	97.4	97.3	100.0	424	5,282	8,120	145,534	8,847	0	759 (0.2:1)	3,631 (1.2:5)	682
Average		48.9	79.7	73.8	100.0	475	4,187	6,627	463,173	14,272	1	2,705 (6.1:4)	3,636 (10.3:5)	541

is much lower than that of the standard formulation. However, with the addition of violated inequalities (8), (9), and (19) to the standard formulation, the root integrality gap is almost closed. Consequently, the number of nodes explored is smaller for the branch-and-cut algorithm. We observe that the tighter the inventory upper bounds, the larger the integrality gap is and the longer the solution time is for the compact formulation. Also, as the ratio of order fixed cost to variable inventory cost (f) increases, the initial integrality gap of the standard formulation as well as the number of explored nodes and the elapsed time increase for the branch-and-cut algorithm. The branch-and-cut algorithm is faster than the branch-and-bound algorithm for the compact formulation for smaller f .

In summary, these computational experiments show that:

1. Inequalities that do not take into account inventory capacities result in a poor performance of the branch-and-cut algorithm;

2. Similarly, inequalities that do not take into account inventory fixed-charge variables also result in a poor performance of the algorithm;

3. However, inequalities that exploit inventory bounds as well as fixed costs are very effective in strengthening the LP relaxations of the lot-sizing problem with inventory bounds and fixed costs consistently for a wide range of cost and capacity parameters;

4. Consequently, they are very useful in solving the problem efficiently.

6. Concluding Remarks

In this paper, we study the facial structure of the polyhedron of lot sizing with inventory upper bounds and fixed costs (LSBI). We first define facet-defining inequalities for the special case with linear inventory costs and then extend them to incorporate inventory fixed-charge variables. We give polynomial time separation algorithms

Table 6. LSBI with fixed and linear inventory costs, compact formulation.

f	c	Compact			Cap				
		gap	nodes	time	gap	rgap	cuts	nodes	time
1,000	2	2.0	12,199	765	19.3	0.1	7,112	9	66
	5	1.7	4,649	401	20.0	0.1	3,806	1	40
	10	1.2	1,126	127	20.5	0.4	10,192	1	17
	20	1.0	136	44	21.4	0.1	4,315	1	16
2,000	2	1.9	10,637	791	22.9	0.3	10,128	9	138
	5	1.7	4,417	396	24.2	0.1	4,472	13	131
	10	1.4	1,297	156	25.3	0.1	8,690	1	71
	20	1.2	278	60	26.6	0.1	13,479	2	61
5,000	2	1.2	6,368	641	26.0	0.2	4,956	23	414
	5	1.6	4,201	417	28.5	0.3	5,872	47	560
	10	1.5	701	119	30.0	0.3	5,169	17	296
	20	0.9	129	51	31.4	0.3	20,754	2	105
Average		1.4	3,845	331	24.7	0.2	8,245	10	159

and a linear programming formulation of LSBI under the Wagner-Whitin nonspeculative cost structure. The computational experiments suggest that the inequalities are very effective in solving the lot-sizing problem with inventory upper bounds and fixed costs.

The inequalities described here and extensions of them may be effective in solving more complicated production/order and inventory-planning problems that contain inventory upper bounds and fixed costs as a substructure. Other classes of strong inequalities for LSBI are described in Küçükyavuz (2004). Currently we are exploring how the results for LSBI can be used for solving general capacitated fixed-charge network flow problems.

Appendix. Strength of the Inequalities

In this section, we give conditions under which the proposed inequalities are facet defining. We introduce the following notation, which will be used throughout this section: For $1 \leq k \leq l \leq n$ and nonempty $S \subseteq [k, l]$, let

$$\begin{aligned} s &= \min\{t \in S\}, \\ s' &= \max\{t \in S\}, \\ \hat{s} &= \min\{t \in [s, l+1] \setminus S\}, \text{ and} \\ \tilde{s} &= \max\{t \in S: t \leq p\} \quad (\tilde{s} = k - 1 \text{ if } S \cap [k, p] = \emptyset). \end{aligned}$$

Also, let e_t and r_t be the unit vectors corresponding to x_t and y_t for $t \in [1, n]$, respectively, q_t be the unit vector corresponding to z_t for $t \in [0, n]$, and r_0 and v_n be the unit vectors for i_0 and i_n . Finally, let $\epsilon > 0$ be an infinitesimally small number.

Under Assumptions 1 and 2, it is easy to check that the dimension of \mathcal{P} is $2n + 1$.

PROPOSITION 12. *Inequality (1) is facet defining for \mathcal{P} if and only if*

1. if $l < n$, then $S \neq \emptyset$,
2. $u_s > d_{(s+1)l}$, where $s = \min\{t \in S\}$,
3. $d_{tl} > 0$ for some $t \in S$ and if $d_{tl} = 0$ for $t \leq l$, then $t \in S$,
4. $u_{s-1} > d_{s(\hat{s}-1)}$, where $\hat{s} = \min\{t \in [s, l+1] \setminus S\}$, and
5. if $l < n$, then $u_{s'} \geq d_{(s'+1)(l+1)}$, where $s' = \max\{t \in S\}$.

PROOF (Necessity). 1. Because for $l < n$, $d_{l+1}(1 - x_{l+1}) \leq i_l$ and $x_{l+1} \leq 1$ dominate inequality (1) with $S = \emptyset$, we must have $S \neq \emptyset$.

2. Assume for contradiction that $u_s \leq d_{(s+1)l}$. Then, adding the $(l, S \setminus \{s\})$ inequality, $\sum_{t \in S \setminus \{s\}} y_t \leq \sum_{t \in S \setminus \{s\}} d_{tl}x_t + i_l$, and the capacity inequality $y_s \leq (d_s + u_s)x_s$ gives an inequality at least as strong as (1) because $d_s + u_s \leq d_{sl}$. Therefore, from Assumption 3, we must have $u_t > d_{(t+1)l}$ for all $t \in [s, l-1]$.

3. If $d_{tl} = 0$ for some $t \in [s, l] \setminus S$, then the $(l, S \cup \{t\})$ inequality dominates (1). If $d_{tl} = 0$ for all $t \in S$, then inequality (1) is implied by the aggregated demand constraint $i_{s-1} + \sum_{t \in [s, l]} y_t = i_l$.

4. If $u_{s-1} < d_{s(\hat{s}-1)}$, then there exists a period $j \in [s+1, \hat{s}-1]$ such that $u_{s-1} < d_{sj}$. Then, the cut-set inequality

$$\sum_{t \in [s, j]} x_t \geq 1 \quad (36)$$

is valid. Now, inequality (1),

$$\sum_{t \in [s, j]} y_t \leq \sum_{t \in [s, j]} d_{tj}x_t + i_j,$$

and the flow balance inequality

$$i_j + \sum_{t \in S \cap [j+1, l]} y_t \leq d_{(j+1)l} + i_l,$$

imply

$$\begin{aligned} \sum_{t \in S} y_t &\leq \sum_{t \in [s, j]} d_{tj}x_t + d_{(j+1)l} + i_l \\ &\leq \sum_{t \in [s, j]} d_{tj}x_t + \sum_{t \in [s, j]} d_{(j+1)l}x_t + i_l \leq \sum_{t \in S} d_{tl}x_t + i_l, \end{aligned}$$

where the second-to-last inequality follows from inequality (36).

On the other hand, if $u_{s-1} = d_{s(\hat{s}-1)}$, then summing inequality (3),

$$i_{s-1} + \sum_{t \in S} y_t \leq u_{s-1} + \sum_{t \in S} \min\{d_{st} - u_{s-1}, d_{tl}\}x_t + i_l,$$

inequality (1)

$$\sum_{t \in [s, \hat{s}-1]} y_t \leq \sum_{t \in [s, \hat{s}-1]} d_{t(\hat{s}-1)}x_t + i_{\hat{s}-1},$$

and the balance equality

$$i_{\hat{s}-1} + d_{s(\hat{s}-1)} = i_{s-1} + \sum_{t \in [s, \hat{s}-1]} y_t,$$

we obtain

$$\sum_{t \in S} y_t \leq \sum_{t \in S} d_{tl}x_t + i_l.$$

5. Because the facet induced by (1) is different from $x_{l+1} = 1$, it contains a point (x, y, i) such that $x_{l+1} = 0$ and, by feasibility, $i_l \geq d_{l+1}$. Because this point satisfies (1) at equality, $y_j = d_{j(l+1)}$ for some $j \in S$, which, by feasibility, implies that $u_{s'} \geq d_{(s'+1)(l+1)}$.

(Sufficiency). To prove sufficiency, we exhibit $2n + 1$ affinely independent points on the face defined by (1). For simplicity, we represent the points in the variable space $(i_0, (y, x), i_n)$. (The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities.) Let e_t and r_t be the unit vectors corresponding to x_t and y_t for $t \in [1, n]$, respectively, and r_0 and v_n be the unit vectors for i_0 and i_n . If $\hat{s} \leq l$, then consider the point

$$\begin{aligned} w_0 &= \sum_{t \in [1, s-2]} (d_t r_t + e_t) + d_{(s-1)(\hat{s}-1)} r_{s-1} + e_{s-1} \\ &\quad + d_{\hat{s}l} r_{\hat{s}} + e_{\hat{s}} + \sum_{t \in [l+1, n]} (d_t r_t + e_t); \end{aligned}$$

otherwise, consider the point

$$w_0 = \sum_{t \in [1, s-2]} (d_t r_t + e_t) + d_{(s-1)(\hat{s}-1)} r_{s-1} + e_{s-1} + \sum_{t \in [l+1, n]} (d_t r_t + e_t)$$

on the face defined by (1). First, we exhibit $2|S|$ affinely independent points. For each $j \in S$, consider the points

$$w_j = \begin{cases} w_0 - d_{j(\hat{s}-1)} r_{s-1} + d_{jl} r_j + e_j - d_{\hat{s}l} r_{\hat{s}} - e_{\hat{s}} & \text{if } j \leq \hat{s} \leq l, \\ w_0 - d_{j(\hat{s}-1)} r_{s-1} + d_{jl} r_j + e_j & \text{if } j \leq \hat{s} = l + 1, \\ w_0 - d_{jl} r_{\hat{s}} + d_{jl} r_j + e_j & \text{otherwise,} \end{cases}$$

$$\bar{w}_j = w_j + \epsilon r_j + \epsilon v_n.$$

Next, consider the following $2(l - s + 1 - |S|)$ points. For $j = \hat{s} \leq l$, let

$$w_j = w_s + e_{\hat{s}} \quad \text{and} \quad \bar{w}_j = w_0 - \epsilon r_{\hat{s}} + \epsilon r_0.$$

For each $j \in [\hat{s} + 1, l] \setminus S$, consider the two points

$$w_j = w_0 + e_j \quad \text{and} \quad \bar{w}_j = w_j + d_{jl} r_j - d_{jl} r_{\hat{s}}.$$

Finally, for each $j \in [1, s - 1] \cup [l + 1, n]$, let

$$w_j = \begin{cases} w_s - e_j - d_j r_j + d_j r_{s'} & \text{if } j = l + 1, \\ w_s - e_j - d_j r_j + d_j r_{j-1} & \text{otherwise,} \end{cases} \quad \text{and} \\ \bar{w}_j = \begin{cases} w_0 + \epsilon r_j + \epsilon v_n & \text{if } j \geq l + 1, \\ w_0 + \epsilon r_j - \epsilon r_{\hat{s}} & \text{if } j \leq s - 1 \text{ and } \hat{s} \leq l, \\ w_0 - \epsilon r_j + \epsilon r_0 & \text{if } j = s - 1 \text{ and } \hat{s} = l + 1, \\ w_0 + \epsilon r_j - \epsilon r_{s-1} & \text{if } j \leq s - 2 \text{ and } \hat{s} = l + 1. \end{cases}$$

These points are affinely independent feasible points satisfying (1) at equality (Küçükyavuz 2004). \square

PROPOSITION 13. *If $u_s \geq d_{(s+1)l}$, then inequality (3) is facet defining for \mathcal{P} if and only if*

1. $u_{k-1} < d_{kl}$;
2. $u_{\tilde{s}} > d_{(\tilde{s}+1)l}$, where $\tilde{s} = \max\{t \in S : t \leq p\}$ ($\tilde{s} = k - 1$ if $S \cap [k, p] = \emptyset$);
3. if $u_{s-1} < d_{sp}$, then either $S = [s, l] = [s, p]$ or $\hat{s} \in [s, p]$;
4. if $d_{tl} = 0$ for some $t \in [k, l]$, then $t \in S$;
5. if $k > 1$, then $u_{k-2} \geq d_{(k-1)(s-1)}$;
6. if $u_{k-1} = d_{k(p-1)}$, then $s < p$, else $s \leq p$;
7. if $l < n$, then $u_{s'} \geq d_{(s'+1)(l+1)}$; and
8. if $u_{k-1} = d_k + u_k$, then $k \in S$.

PROOF. Note that, by Assumption 3, $u_s \geq d_{(s+1)l}$ if and only if $d_{kt} + u_t \geq d_{kl}$ for all $t \in S$. Then, inequality (3) simplifies as

$$i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} \min\{d_{kl} - u_{k-1}, d_{tl}\} x_t + i_l. \quad (37)$$

If $u_s < d_{(s+1)l}$, then (3) is stronger than (37).

(Necessity). 1. If $u_{k-1} = d_{kl}$, then inequality (3) is implied by the aggregated flow balance equality

$$i_{k-1} + \sum_{j \in [k, l]} y_j = d_{kl} + i_l.$$

2. If $u_{\tilde{s}} \leq d_{(\tilde{s}+1)l}$, then inequality (3) is dominated by (5),

$$i_{k-1} + \sum_{j \in S: j \leq p} y_j \leq u_{k-1} + \sum_{j \in S: j \leq p} (d_{k\tilde{s}} - u_{k-1} + u_{\tilde{s}}) x_j,$$

and inequality (1),

$$\sum_{j \in S: j > p} y_j \leq \sum_{j \in S: j > p} d_{jl} x_j + i_l.$$

Therefore, we must have $u_{\tilde{s}} > d_{(\tilde{s}+1)l}$. Note that, from Assumption 3, this condition implies that $u_t > d_{(t+1)l}$ for all $t \in [\tilde{s}, l]$.

3. Suppose that the condition is not true. Either $S = [s, p]$ and $k < s \leq p < l$, or $[s, p] \subset S$. Because $u_{s-1} < d_{sp}$, it follows that

$$\sum_{t \in [s, p]} x_t \geq 1.$$

Multiplying both sides of this inequality with $d_{kl} - u_{k-1}$ and adding $d_{kl} + i_l = i_{k-1} + \sum_{t \in [k, l]} y_t$, we obtain

$$\sum_{t \in [k, l]} y_t \leq \sum_{t \in [s, p]} (d_{kl} - u_{k-1}) x_t + (u_{k-1} - i_{k-1}) + i_l,$$

which is stronger than (3) in either case.

4. Note that $d_{tl} > 0$ for all $t \in [k, p]$ because we must have $0 < u_{k-1} < d_{kp}$. If $d_{tl} = 0$ for some $t \in [p + 1, l]$ and $t \notin S$, then inequality (3), where S augmented with t , is stronger.

5. Because the facet defined by (3) is different from $x_{k-1} = 1$, there is a point (x, y, i) on the face with $x_{k-1} = 0$. Because (3) is satisfied at equality, we have $i_{k-1} \geq d_{k(s-1)}$. As $y_{k-1} = 0$, this implies that $u_{k-2} \geq i_{k-2} \geq d_{(k-1)(s-1)}$.

6. If $s > p$, then inequality (3) is obtained by adding $i_{k-1} \leq u_{k-1}$ and the (l, S) inequality. Therefore, we must have $s \leq p$. However, if $u_{k-1} = d_{k(p-1)}$, then $d_{kl} - u_{k-1} = d_{pl}$ and by the same argument, we need $s < p$. Note that this condition implies that $S \neq \emptyset$.

7. Follows from the same argument for Item 5 of Proposition 12.

8. Suppose that $u_{k-1} = d_k + u_k$ and $k \notin S$. Then, inequality

$$i_k + \sum_{t \in S} y_t \leq u_k + \sum_{t \in S} \min\{d_{(k+1)l} - u_k, d_{tl}\} x_t + i_l$$

is stronger than (3). This can be seen by subtracting the flow balance equality

$$i_{k-1} + y_k = d_k + i_k$$

from (3).

(Sufficiency). We define $2n + 1$ affinely independent points on the face defined by (3). For simplicity, we represent the points in the variable space $(i_0, (y, x), i_n)$. (The values of the variables i_t for $t \in [1, n - 1]$ can be obtained via the flow balance equalities.) For consistency of notation, let d_0 be zero and e_0 be the zero vector. Consider the point $w_0 = \sum_{t \in [1, k-2]} (d_t r_t + e_t) + \sum_{t \in [l+1, n]} (d_t r_t + e_t) + (d_{k-1} + u_{k-1})r_{k-1} + e_{k-1}$. Note that w_0 is not feasible because $u_{k-1} < d_{kl}$. We perturb w_0 to define feasible points.

First, consider the following $2|S|$ points. For each $j \in S$, if $j \leq p$, then let

$$w_j = w_0 + e_j + (d_{kl} - u_{k-1})r_j;$$

otherwise let

$$w_j = \begin{cases} w_0 + e_j + d_{jl}r_j + (d_{k(j-1)} - u_{k-1})r_{\hat{s}} + e_{\hat{s}} & \text{if } \hat{s} \leq p, \\ w_0 + e_j + d_{jl}r_j + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + e_{s-1} \\ \quad + d_{\hat{s}(j-1)}r_{\hat{s}} + e_{\hat{s}} & \text{if } s > k, \hat{s} > p, \\ w_0 + e_j + d_{jl}r_j + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + d_{\hat{s}(j-1)}r_{\hat{s}} + e_{\hat{s}} \\ \quad \text{if } s = k, \hat{s} > p. \end{cases}$$

Also, for $j \in S$, let

$$\bar{w}_j = \begin{cases} w_j + \epsilon r_j + \epsilon v_n & \text{if } j \geq \tilde{s}, \\ w_j + \epsilon r_j - \epsilon r_{k-1} & \text{otherwise.} \end{cases}$$

For $j = k - 1$ and $k > 1$, let

$$w_j = w_0 - e_{k-1} - (d_{k-1} + u_{k-1})r_{k-1} \\ + d_{(k-1)(s-1)}r_{k-2} + d_{sl}r_s + e_s \quad \text{and}$$

$$\bar{w}_j = w_{\tilde{s}} - \epsilon r_{k-1} + \epsilon r_{\tilde{s}}.$$

For $j \in [k, l] \setminus S$ consider

$$w_j = w_s + e_j \quad \text{and}$$

$$\bar{w}_j = \begin{cases} w_0 + (d_{kl} - u_{k-1})r_j + e_j \\ \quad \text{if } j \in [\hat{s}, p] \text{ or } j = s - 1, \hat{s} > p, \\ \bar{w}_{\hat{s}} + \epsilon r_j + e_j - \epsilon r_{\hat{s}} & \text{if } j \in [p + 1, l] \setminus \{\hat{s}\}, \\ \bar{w}_{\hat{s}} + \epsilon r_j + e_j - \epsilon r_{\hat{s}} & \text{if } j \in [k, \hat{s} - 1] \text{ and } \hat{s} \leq p, \\ w_{s-1} + \epsilon r_j + e_j - \epsilon r_{s-1} & \text{if } j \in [k, s - 2] \text{ and } \hat{s} \geq p, \\ w_0 + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + e_{s-1} + d_{\hat{s}l}r_{\hat{s}} + e_{\hat{s}} \\ \quad \text{if } j = \hat{s} > p \text{ and } s > k, \\ w_0 + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + d_{\hat{s}l}r_{\hat{s}} + e_{\hat{s}} \\ \quad \text{if } j = \hat{s} > p \text{ and } s = k. \end{cases}$$

For $j \in [l + 1, n]$ and $l < n$, consider

$$w_j = \begin{cases} w_{s'} - e_j - d_j r_j + d_j r_{s'} & \text{if } j = l + 1, \\ w_{s'} - e_j - d_j r_j + d_j r_{j-1} & \text{if } j \in [l + 2, n], \end{cases} \quad \text{and}$$

$$\bar{w}_j = w_{s'} + \epsilon r_j + \epsilon v_n.$$

For $j \in [1, k - 2]$ and $k > 2$, let

$$w_j = w_{s'} - e_j - d_j r_j + d_j r_{j-1} \quad \text{and} \quad \bar{w}_j = w_{s'} + \epsilon r_j - \epsilon r_{k-1}.$$

Finally, let

$$\bar{w}_0 = \begin{cases} w_s + \epsilon r_0 - \epsilon r_{k-1} & \text{if } k > 1, \\ w_{\tilde{s}} - \epsilon r_{k-1} + \epsilon r_{\tilde{s}} & \text{if } k = 1. \end{cases}$$

The $2n + 1$ points given above are affinely independent points on the face (3) (Küçükyavuz 2004). \square

PROPOSITION 14. Let $S \subseteq [k, l]$ for $1 \leq k \leq l \leq n$. If $u_s \geq d_{(s+1)l} + u_l$, then inequality (5) is facet defining for \mathcal{P} if and only if

1. $u_{k-1} < d_{kl} + u_l$;
2. $S \subseteq [k, q]$;
3. if $u_{s-1} < d_{sq}$, then $S \neq [s, q]$;
4. if $k > 1$, then $u_{k-2} \geq d_{(k-1)(s-1)}$ if $S \neq \emptyset$ and $u_{k-2} \geq d_{k-1} + u_{k-1}$ if $S = \emptyset$; and
5. if $u_{k-1} = d_k + u_k$, then $k \in S$.

PROOF. Note that by Assumption 3, $u_s \geq d_{(s+1)l} + u_l$ if and only if $d_{kt} + u_t = d_{kl} + u_l$ for all $t \in S$. Then, inequality (3) simplifies as

$$i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} (d_{kl} - u_{k-1} + u_l)x_t. \quad (38)$$

If $u_s < d_{(s+1)l} + u_l$, then (5) is stronger than (38).

(Necessity). 1. From Assumption 3, $u_{k-1} \leq d_{kl} + u_l$. If $u_{k-1} = d_{kl} + u_l$, then inequality (5) is implied by the aggregated flow balance equality

$$i_{k-1} + \sum_{t \in [k, l]} y_t = d_{kl} + i_l \leq d_{kl} + u_l.$$

2. The statement holds trivially if $q = l$. Otherwise, from the definition of q , we have $d_{kl} - u_{k-1} + u_l \geq d_{(q+1)l} + u_l \geq d_t + u_t$ for all $t \in [q + 1, l]$. Then, if $t \in S \cap [q + 1, l]$, the inequality is dominated by the one with $S' = S \setminus \{t\}$ and $y_t \leq (d_t + u_t)x_t$.

3. Suppose that $u_{s-1} < d_{sq}$ and $S = [s, q]$. Then,

$$\sum_{t \in [s, q]} x_t \geq 1$$

is valid. Multiplying both sides of this inequality with $d_{kq} - u_{k-1} + u_q$, and adding $i_q \leq u_q$ and $i_{k-1} + \sum_{t \in [k, q]} y_t = d_{kq} + i_q$, gives

$$i_{k-1} + \sum_{t \in [k, q]} y_t \leq u_{k-1} + \sum_{t \in [s, q]} (d_{kq} - u_{k-1} + u_q)x_t. \quad (39)$$

However, from Condition 2 and Assumption 3, we have $u_q = d_{(q+1)l} + u_l$. Thus, (39) equals

$$i_{k-1} + \sum_{t \in [k, q]} y_t \leq u_{k-1} + \sum_{t \in [s, q]} (d_{kl} - u_{k-1} + u_l)x_t.$$

4. If $S \neq \emptyset$, the proof is the same as for Condition 5 of Proposition 13. If $S = \emptyset$ and $u_{k-2} < d_{k-1} + u_{k-1}$, then $i_{k-1} \leq u_{k-1}$ is dominated by the valid inequality $(u_{k-1} + d_{k-1} - u_{k-2})(1 - x_{k-1}) + i_{k-1} \leq u_{k-1}$.

5. If $u_{k-1} = d_k + u_k$ and $k \notin S$, then adding inequality (5),

$$i_k + \sum_{t \in S} y_t \leq u_k + \sum_{t \in S} (d_{(k+1)l} - u_k + u_t)x_t,$$

and the flow balance equality $i_{k-1} + y_k = d_k + i_k$, we obtain a stronger inequality.

(Sufficiency). By Assumption 3, we have $u_q = d_{(q+1)l} + u_l$, and by Condition 2 we have $S \subseteq [k, q]$ where $q \leq l$. Therefore, we assume below that $l = q$. We give $2n + 1$ affinely independent points on the face (5). For simplicity, we represent the points in the variable space $(i_0, (y, x), i_n)$. (The values of the variables i_t for $t \in [1, n - 1]$ can be obtained via the flow balance equalities.) Let $w_0 = \sum_{j \in [1, k-1]} (e_j + d_j r_j) + \sum_{j \in [l+1, n]} (e_j + \bar{d}_j r_j) + e_{k-1} + (d_{k-1} + u_{k-1})r_{k-1}$, where \bar{d}_j is the remaining demand for period j when $i_l = u_l$ and there is no production in $[l, j]$, i.e., $\bar{d}_j = d_j - (u_l - d_{(l+1)(j-1)})^+$. (Let d_0 be zero and e_0 be the zero vector for $k = 1$.) Note that w_0 is not feasible because $u_{k-1} < d_{kl} + u_l$. Similarly, let d'_j be the remaining demand for period j when $i_{k-1} = u_{k-1}$ and there is no production in $[k, j]$, i.e., $d'_j = d_j - (u_{k-1} - d_{k(j-1)})^+$. Let $l = k$ and $s = k - 1$ if $S = \emptyset$. Consider the following feasible point of \mathcal{P} :

$$\tilde{w} = \begin{cases} w_0 + (d_{kl} - u_{k-1})r_{\hat{s}} + e_{\hat{s}} + \sum_{j \in [l+1, n]} (d_j - \bar{d}_j)r_j & \text{if } S \neq \emptyset \text{ and } \hat{s} \leq l, \\ w_0 + (d_{kl} - u_{k-1})r_{s-1} + e_{s-1} + \sum_{j \in [l+1, n]} (d_j - \bar{d}_j)r_j & \text{if } S \neq \emptyset, s > k, \hat{s} > l, \\ w_0 + \sum_{j \in [l+1, n]} (d'_j - \bar{d}_j)r_j & \text{if } S \neq \emptyset, s = k, \hat{s} > l, \\ w_0 + \sum_{j \in [l+1, n]} (d'_j - \bar{d}_j)r_j & \text{if } S = \emptyset. \end{cases}$$

For $j \in S$, consider

$$w_j = w_0 + e_j + (d_{kl} + u_l - u_{k-1})r_j \quad \text{and} \\ \bar{w}_j = w_j + \epsilon r_j - \epsilon r_{k-1}.$$

For $j = k - 1$ and $k > 1$ if $S \neq \emptyset$, then consider

$$w_j = w_0 - e_{k-1} - (d_{k-1} + u_{k-1})r_{k-1} + d_{(k-1)(s-1)}r_{k-2} \\ + (u_l + d_{sl})r_s + e_s,$$

otherwise consider

$$w_j = \tilde{w} - e_{k-1} - (d_{k-1} + u_{k-1})r_{k-1} + (d_{k-1} + u_{k-1})r_{k-2}.$$

Also, let

$$\bar{w}_{k-1} = \tilde{w} - \epsilon r_j + \epsilon r_0.$$

For $j \in [k, l] \setminus S$, consider

$$w_j = \begin{cases} w_s + e_j & \text{if } S \neq \emptyset, \\ w_0 + e_j + \sum_{t \in [l+1, n]} (d'_t - \bar{d}_t) & \text{if } S = \emptyset, \end{cases} \quad \text{and} \\ \bar{w}_j = \begin{cases} w_0 + e_j + (d_{kl} + u_l - u_{k-1})r_j & \text{if } j \in [s + 1, l], \\ \bar{w}_{\hat{s}} + \epsilon r_j + e_j - \epsilon r_{\hat{s}} & \text{if } j \in [k, s] \text{ and } \hat{s} \leq l, \\ \tilde{w} + \epsilon r_j + e_j - \epsilon r_{s-1} & \text{if } j \in [k, s - 2] \text{ and } \hat{s} = l + 1, \\ \tilde{w} + \epsilon r_j + \epsilon v_n & \text{if } j = s - 1 \text{ and } \hat{s} = l + 1. \end{cases}$$

For $j \in [1, k - 2]$ and $k > 2$, let

$$w_j = \tilde{w} - e_j - d_j r_j + d_j r_{j-1} \quad \text{and} \quad \bar{w}_j = \tilde{w} + \epsilon r_j - \epsilon r_{k-1}.$$

For $j = l + 1$, let

$$w_j = \begin{cases} w_s - e_j & \text{if } S \neq \emptyset, \\ \bar{w}_l - e_j & \text{otherwise,} \end{cases} \quad \text{and} \\ \bar{w}_j = \tilde{w} + \epsilon r_j + \epsilon v_n.$$

For $j \in [l + 2, n]$, let $w_j = \tilde{w} - e_j - d_j r_j + d_j r_{j-1}$ and $\bar{w}_j = \tilde{w} + \epsilon r_j + \epsilon v_n$. These $2n + 1$ points are affinely independent points satisfying inequality (5) at equality (Küçükyavuz 2004). \square

Under Assumptions 1 and 2, it is easy to verify that the dimension of \mathcal{Q} is $3n + 2$.

PROPOSITION 15. Inequality (8) is facet defining for \mathcal{Q} under the conditions of Proposition 12 if $u_{k-1} > d_{kl}$ and either $S = [k, b_k] = [k, l]$ or $[k, b_k] \setminus S \neq \emptyset$, where $b_k = \min\{t \in [k, l]: d_t > 0\}$.

PROOF. For simplicity, we represent the points in the variable space $(i_0, (y, x, z), i_n)$. (The values of the variables i_t for $t \in [1, n - 1]$ can be obtained via the flow balance equalities.) Let q_t be the unit vector corresponding to z_t for $t \in [0, n]$. For the case that $k = 0$, we use $2n + 1$ points described in Proposition 12 with $q_j = 1$ for all $j \in [0, n]$. For $j \in [0, s - 2] \cup [l + 1, n]$, consider $\tilde{w}_j = w_s + \sum_{t \in [0, n] \setminus \{j\}} q_t$. For $j \in [s - 1, l]$, let $\tilde{w}_j = w_j + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $j = n$. If $j < n$, then consider $\tilde{w}_j = w_{j+1} + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $\{j + 1\} \in S$, $\tilde{w}_j = w_0 + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $j + 1 = \hat{s}$, and $\tilde{w}_j = \bar{w}_{j+1} + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $\{j + 1\} \notin S$ and $j + 1 \neq \hat{s}$. These $3n + 2$ points are affinely independent.

Now for $k > 0$, let w_j for all $j \in [1, n]$ be as described in the sufficiency proof of Proposition 12. Let $\hat{w}_{k-1} = \sum_{t \in [1, k-2] \cup [l+1, n]} (d_t r_t + e_t) + d_{(k-1)l} r_{k-1} + e_{k-1} + \sum_{t \in [k-1, l-1]} q_t$, and $\bar{w}_{k-1} = \hat{w}_{k-1} + \epsilon r_{k-1} + \epsilon v_n + \sum_{t \in [l, n]} q_t$. Also, if $k > 1$, let $\bar{w}_{k-1} = \hat{w}_k - d_{k-1} r_{k-1} - e_{k-1} + d_{k-1} r_{k-2} + q_{k-2}$. If $[k, b_k] \setminus S \neq \emptyset$, then let $\bar{s} \in [k, b_k] \setminus S$. For $j \in S$, consider the points

$$\hat{w}_j = \begin{cases} \sum_{t \in [1, k-1] \cup [l+1, n]} (d_t r_t + e_t) + d_j r_j + e_j + \sum_{t \in [j, l-1]} q_t & \text{if } j \leq b_k, \\ \sum_{t \in [1, k-1] \cup [l+1, n]} (d_t r_t + e_t) + d_j r_j + e_j + d_{\bar{s}(j-1)} r_{\bar{s}} + e_{\bar{s}} & \text{otherwise,} \end{cases}$$

$$\bar{w}_j = \hat{w}_j + \epsilon r_j + \epsilon v_n + \sum_{t \in [l, n]} q_t, \quad \text{and}$$

$$\tilde{w}_j = \begin{cases} \hat{w}_{j+1} + q_j & \text{if } j < n, \\ \hat{w}_j + q_j & \text{otherwise.} \end{cases}$$

For $j \in [k, b_k] \setminus S$, consider

$$\hat{w}_j = \sum_{t \in [1, k-1] \cup [l+1, n]} (d_t r_t + e_t) + d_{j_l} r_j + e_j + \sum_{t \in [j, l-1]} q_t,$$

and for $j \in [b_k + 1, l] \setminus S$, consider

$$\hat{w}_j = \sum_{t \in [1, k-1] \cup [l+1, n]} (d_t r_t + e_t) + d_{j_l} r_j + e_j + d_{\bar{s}(j-1)} r_{\bar{s}} + e_{\bar{s}} + \sum_{t \in [\bar{s}, j-2] \cup [j, l-1]} q_t.$$

Also, for $j \in [k, l] \setminus S$, consider

$$\bar{w}_j = \hat{w}_{k-1} + e_j, \quad \text{and}$$

$$\tilde{w}_j = \begin{cases} \hat{w}_{j+1} + q_j & \text{if } j < n, \\ \hat{w}_j + q_j & \text{otherwise.} \end{cases}$$

For $j \in [1, k-2] \cup [l+1, n]$, consider

$$\hat{w}_j = \begin{cases} \hat{w}_{s'} - d_j r_j - e_j + d_j r_{s'} + q_l & \text{if } j = l + 1, \\ \hat{w}_s - d_j r_j - e_j + d_j r_{j-1} + q_{j-1} & \text{otherwise,} \end{cases}$$

$$\bar{w}_j = \begin{cases} \hat{w}_{k-1} + \epsilon r_j - \epsilon r_{k-1} + \sum_{t \in [j, k-2]} q_t & \text{if } j \leq k - 2, \\ \hat{w}_{k-1} + \epsilon r_j + \epsilon v_n + \sum_{t \in [j, n]} q_t & \text{otherwise,} \end{cases}$$

$$\tilde{w}_j = \hat{w}_{k-1} + q_j.$$

Finally, for $k > 1$, consider the following two points:

$$\bar{w}_0 = \hat{w}_{k-1} + \epsilon r_0 - \epsilon r_{k-1} + \sum_{t \in [0, k-2]} q_t, \quad \text{and}$$

$$\tilde{w}_0 = \hat{w}_{k-1} + q_0.$$

These $3n + 2$ points are affinely independent feasible points that satisfy (8) at equality (Küçükyavuz 2004). \square

PROPOSITION 16. *Inequality (9) is facet defining for \mathcal{Q} if $\{k-1\} \cup ([k, p-1] \cap S) \subseteq T$ and the conditions of Proposition 13 are satisfied.*

PROOF. For simplicity, we represent the points in the variable space $(i_0, (y, x, z), i_n)$. The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities. Consider the $2n + 1$ affinely independent points described in Proposition 13 with $q_t = 1$ for all $t \in [0, n]$. For each $j \in [0, k-2] \cup [l, n]$, consider the point $\hat{w}_j = w_s + \sum_{t \in [0, n] \setminus \{j\}} q_t$, where w_s is as defined in Proposition 13. For each $j \in [p, l-1]$, if $\hat{s} \leq p$, then consider

$\hat{w}_j = \sum_{t \in [1, k-2] \cup [l+1, n]} (d_t r_t + e_t) + (d_{k_j} - u_{k-1}) r_{\hat{s}} + e_{\hat{s}} + \sum_{t \in [k-1, l-1] \setminus \{j\}} q_t + (d_{k-1} + u_{k-1}) r_{k-1} + e_{k-1} + d_{(j+1)l} r_{j+1} + e_{j+1}$, otherwise let $\hat{w}_j = \sum_{t \in [1, k-2] \cup [l+1, n]} (d_t r_t + e_t) + (d_{k(\hat{s}-1)} - u_{k-1}) r_{s-1} + e_{s-1} + d_{\hat{s}} r_{\hat{s}} + e_{\hat{s}} + \sum_{t \in [k-1, l-1] \setminus \{j\}} q_t + (d_{k-1} + u_{k-1}) r_{k-1} + e_{k-1} + d_{(j+1)l} r_{j+1} + e_{j+1}$. For each $j \in [k-1, p-1]$ with $s(j) \leq l$, letting $t(j) = \max\{t \in T: t \leq j\}$, consider $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t(j)+1, s(j)-1] \cup [l+1, n]} (d_t r_t + e_t) + \sum_{t \in [k-1, l-1] \setminus (\{t(j), s(j)\} \cap (T \cup \{j\}))} q_t + d_{(k-1)t(j)} r_{k-1} + e_{k-1} + d_{s(j)l} r_{s(j)} + e_{s(j)}$. For each $j \in [k-1, p-1]$ with $s(j) > l$, consider $\hat{w}_j = \sum_{t \in [1, k-2]} (d_t r_t + e_t) + d_{(k-1)t(j)} r_{k-1} + e_{k-1} + \sum_{t \in [k-1, l-1] \setminus (\{t(j), s(j)\} \cap (T \cup \{j\}))} q_t + \sum_{t \in [t(j)+1, s(j)-1] \cup [l+1, n]} (d_t r_t + e_t)$. These $3n + 2$ points are affinely independent points on the face (9) (Küçükyavuz 2004). \square

PROPOSITION 17. *Inequality (19) is facet defining for \mathcal{Q} if $\{k-1\} \cup ([k, p-1] \cap S) \subseteq T$ and the conditions of Proposition 14 are satisfied.*

PROOF. For simplicity, we represent the points in the variable space $(i_0, (y, x, z), i_n)$. The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities. Consider the $2n + 1$ affinely independent points described in Proposition 14 with $q_j = 1$ for all $j \in [0, n]$. For each $j \in [0, k-2]$, consider $\hat{w}_j = \tilde{w} + \sum_{t \in [0, n] \setminus \{j\}} q_t$, where \tilde{w} is as defined in Proposition 14. For each $j \in [t_\tau, n]$, let $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t_\tau+1, n]} (d_t r_t + e_t) + d_{(k-1)t_\tau} r_{k-1} + e_{k-1} + \sum_{t \in [k-1, t_\tau-1] \cup [t_\tau+1, n] \setminus \{j\}} q_t$. Finally, for $j \in [k-1, t_\tau-1]$, letting $t(j) = \max\{t \in T: t \leq j\}$, if $s(j) \leq l$, then consider $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t(j)+1, s(j)-1]} (d_t r_t + e_t) + \sum_{t \in [l+1, n]} (\bar{d}_t r_t + e_t) + d_{(k-1)t(j)} r_{k-1} + e_{k-1} + \sum_{t \in [k-1, l-1] \setminus (\{t(j), s(j)-1\} \cap (T \cup \{j\}))} q_t + (d_{s(j)l} + u_l) r_{s(j)} + e_{s(j)} + \sum_{t \in [l+1, n]} q_t$, where \bar{d}_t is the effective demand in period t if $i_t = u_t$. If $s(j) > l$, then consider $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t(j)+1, s(j)-1]} (d_t r_t + e_t) + d_{(k-1)t(j)} r_{k-1} + e_{k-1} + \sum_{t \in [k-1, l-1] \setminus (\{t(j), s(j)-1\} \cap (T \cup \{j\}))} q_t + \sum_{t \in [l+1, n]} (d_t r_t + e_t)$. These $3n + 2$ points are affinely independent points on the face (19) (Küçükyavuz 2004). \square

Endnote

1. CPLEX is a trademark of ILOG, Inc.

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